On some aspects of the Deligne-Simpson problem *

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Abstract

The Deligne-Simpson problem in the multiplicative version is formulated like this: *give necessary and sufficient conditions for the choice of the conjugacy classes $C_j \in SL(n, \mathbf{C})$ so that there exist irreducible $(p+1)$-tuples of matrices $M_j \in C_j$ satisfying the equality $M_1 \ldots M_{p+1} = I$.*

We solve the problem for generic eigenvalues in the case when all the numbers $\Sigma_{j,m}(\sigma)$ of Jordan blocks of a given matrix M_j, with a given eigenvalue σ and of a given size m (taken over all j, σ, m) are divisible by $d > 1$. Generic eigenvalues are defined by explicit algebraic inequalities of the form $a \neq 0$. For such eigenvalues there exist no reducible $(p+1)$-tuples.

The matrices M_j are interpreted as monodromy operators of regular linear systems on Riemann's sphere.

Key words: generic eigenvalues, (poly)multiplicity vector, corresponding Jordan normal forms, monodromy operator.

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Contents

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1 Introduction

1.1 Formulation of the problem

In the present paper we consider the multiplicative version of the *Deligne-Simpson problem:*

Give necessary and sufficient conditions for the choice of the conjugacy classes $C_j \in SL(n, \mathbf{C})$ so that there exist irreducible $(p+1)$-tuples of matrices $M_j \in C_j$ such that

$$M_1 \ldots M_{p+1} = I \tag{1}$$

In the additive version the conjugacy classes c_j belong to $sl(n, \mathbf{C})$ and the matrices $A_j \in c_j$ satisfy the condition

$$A_1 + \ldots + A_{p+1} = 0 \tag{2}$$

The matrices A_j and M_j are interpreted respectively as matrices-residua and as monodromy operators of fuchsian linear systems, see the next section. Both versions of the problem were considered in [Ko1] which is the first part of this paper.

The problem (in the multiplicative version) was stated by P.Deligne and C.Simpson obtained the first results towards its solution, see [Si].

Further in the text we consider sometimes C_j (c_j) as conjugacy classes from $GL(n, \mathbf{C})$ (from $gl(n, \mathbf{C})$) instead of $SL(n, \mathbf{C})$ (instead of $sl(n, \mathbf{C})$) because when solving the problem there appear such matrices. The passage from the problem for $M_j \in SL(n, \mathbf{C})$ (or $A_j \in sl(n, \mathbf{C})$) to the one for $M_j \in GL(n, \mathbf{C})$ (or $A_j \in gl(n, \mathbf{C})$) and vice versa is trivial.

We presume that there hold the necessary conditions $\sum_{j=1}^{p+1}\mathrm{Tr}(c_j) = 0$ and $\prod_{j=1}^{p+1}\det(C_j) = 1$. This means that the eigenvalues $\lambda_{k,j}$ ($\sigma_{k,j}$) of the matrices A_j (M_j) satisfy the conditions

$$\sum_{j=1}^{p+1}\sum_{k=1}^{n}\lambda_{k,j}=0 \quad , \quad \prod_{j=1}^{p+1}\prod_{k=1}^{n}\sigma_{k,j}=1$$

Here $k=1,\ldots,n$ and for j fixed the eigenvalues are not presumed distinct. By definition, the *multiplicity* of the eigenvalue $\lambda_{k,j}$ (or $\sigma_{k,j}$) is the number of eigenvalues $\lambda_{i,j}$ (or $\sigma_{i,j}$; j is fixed) equal to it including the eigenvalue itself.

Define as *generic* any set of eigenvalues $\lambda_{k,j}$ or $\sigma_{k,j}$ which satisfy none of the equalities

$$\textstyle\sum_{j=1}^{p+1}\sum_{k\in\Phi_j}\lambda_{k,j} \;=\; 0 \quad , \quad \prod_{j=1}^{p+1}\prod_{k\in\Phi_j}\sigma_{k,j} \;=\; 1 \qquad (\gamma)$$

where the sets Φ_j contain one and the same number κ of indices $(1<\kappa<n)$ for all j. Reducible $(p+1)$-tuples exist only for non-generic eigenvalues – if a $(p+1)$-tuple is block upper-triangular, then the eigenvalues of each diagonal block satisfy some relation (γ).

Definition. Call *Problem (I)* the Deligne-Simpson problem like it is formulated above and *Problem (TC)* the same problem in which the requirement of irreducibility is replaced by the requirement the centralizer of the $(p+1)$-tuple to be trivial.

It is clear that Problem (TC) is weaker than Problem (I) and that for generic eigenvalues the answers to both problems coincide. Part of the results from this paper concern Problem (TC).

1.2 The quantities q, d, ξ and m_0

Definition. A *multiplicity vector (MV)* is a vector whose components are non-negative integers whose sum equals n. We always interpret a MV as the vector of the multiplicities of the eigenvalues of a matrix M_j or A_j. A *polymultiplicity vector (PMV)* is a $(p+1)$-tuple or $(p+2)$-tuple of MVs (depending on whether we deal with a $(p+1)$- or $(p+2)$-tuple of matrices A_j or M_j). A PMV is called *simple* (resp. *non-simple*) if the greatest common divisor q of all the components of all its MVs equals 1 (if not).

Notation. Denote by $\Sigma_{j,m}(\sigma)$ the number of Jordan blocks of size m, of a given matrix M_j and corresponding to its eigenvalue σ. Denote by d the greatest common divisor of all numbers $\Sigma_{j,m}(\sigma)$ (over all σ, j and m).

Remark: The condition $q=1$ implies $d=1$, and $d>1$ implies $q>1$ but the inverse implications are not true. It is true that d divides q and that q divides n.

In [Ko1] we considered the case of generic eigenvalues: the additive version was considered completely, in the multiplicative one we considered only the situation when $d=1$. In the present second part of [Ko1] we consider the multiplicative version for generic eigenvalues and $d>1$ (we call it *the case* $d>1$).

Notation. Let $q>1$. Denote by ξ the product of the eigenvalues $\sigma_{k,j}$ when their multiplicities are reduced q times. Hence, ξ is a root of unity of order q. Set $\xi=\exp(2\pi i m_0/q)$. If $(m_0,q)=1$, then ξ is a primitive root. Another equivalent definition of m_0 is given in 5) of the remarks in Subsection 2.1.

Example: If $p=n=2$ and if each of the three Jordan normal forms consists of one Jordan block of size 2, then there are two possibilities – either $\sigma_{1,1}\sigma_{1,2}\sigma_{1,3}=1$ or $\sigma_{1,1}\sigma_{1,2}\sigma_{1,3}=-1$ (because $\sigma_{1,j}=\sigma_{2,j}$ and $(\sigma_{1,1}\sigma_{1,2}\sigma_{1,3})^2=1$). In the first (in the second) case there are no (there are) irreducible triples of matrices M_j satisfying (1). In the first (in the second) case the eigenvalues are not (are) generic. In both cases one has $q=2$, $d=1$. In the first case $\xi=1$, in the second case $\xi=-1$.

Remark: If $q > 1$ and if ξ is not a primitive root of unity of order q, then the eigenvalues $\sigma_{k,j}$ are not generic. However, ξ can be a primitive root of unity and the eigenvalues can be non-generic.

Example: Let $n = 4$, $p = 3$ and let all Jordan normal forms be diagonal. Let for all j one have $\sigma_{1,j} = \sigma_{2,j} \neq \sigma_{3,j} = \sigma_{4,j}$. Hence, $d = q = 2$. Let $\sigma_{1,1} = \sigma_{1,2} = \sigma_{1,3} = 1$, $\sigma_{1,4} = i$, $\sigma_{3,1} = \sigma_{3,2} = \exp(\pi i/4)$, $\sigma_{3,3} = \exp(-\pi i/8)$, $\sigma_{3,4} = \exp(\pi i/8)$. Hence, $\xi = -1$. One has $\sigma_{1,1}\sigma_{1,2}\sigma_{3,3}\sigma_{3,4} = 1$ which is a non-genericity relation.

In the additive version, when $q > 1$, the eigenvalues $\lambda_{k,j}$ satisfy a non-genericity relation – if their multiplicities are reduced q times, then their sum is 0. Denote this relation by (γ_1).

In the multiplicative version, when ξ is not a primitive root of unity, the eigenvalues $\sigma_{k,j}$ satisfy a non-genericity relation (γ^*) – when their multiplicities are reduced (m_0, q) times, then their product equals 1. The *multiples* of (γ^*) are obtained (by definition) when after the reduction (m_0, q) times the multiplicities are increased s times with $1 \leq s < (m_0, q)$. The multiples are denoted by $s(\gamma^*)$. (The multiples of (γ_1) are defined by analogy.)

Definition. If $q > 1$, if in the multiplicative version $(m_0, q) > 1$ and if the only non-genericity relation satisfied by the eigenvalues $\lambda_{k,j}$ (resp. $\sigma_{k,j}$) is (γ_1) (resp. (γ^*)) and its multiples, then the eigenvalues are called *relatively generic.*

1.3 The results obtained up to now

Definition. Call *Jordan normal form of size* n a family $J^n = \{b_{i,l}\}$ ($i \in I_l$, $I_l = \{1, \ldots, s_l\}$, $l \in L$) of positive integers $b_{i,l}$ whose sum is n. Here L is the set of eigenvalues (all distinct) and I_l is the set of Jordan blocks with eigenvalue l, $b_{i,l}$ is the size of the i-th block with this eigenvalue. We presume that the following inequalities hold (for each l fixed):

$$b_{1,l} \geq b_{2,l} \geq \ldots \geq b_{s_l,l}$$

An $n \times n$-matrix Y has the Jordan normal form J^n (notation: $J(Y) = J^n$) if to its distinct eigenvalues λ_l, $l \in L$, there belong Jordan blocks of sizes $b_{i,l}$.

For a conjugacy class C in $GL(n, \mathbf{C})$ or $gl(n, \mathbf{C})$ denote by $d(C)$ its dimension and for a matrix Y from C set $r(C) := \min_{\lambda \in \mathbf{C}} \mathrm{rk}(Y - \lambda I)$. The integer $n - r(C)$ is the maximal number of Jordan blocks of $J(Y)$ with one and the same eigenvalue. Set $d_j := d(C_j)$ (resp. $d(c_j)$), $r_j := r(C_j)$ (resp. $r(c_j)$). The quantities $r(C)$ and $d(C)$ depend only on the Jordan normal form $J(Y) = J^n$, not on the eigenvalues, so we write sometimes $r(J^n)$ and $d(J^n)$.

The following two conditions are necessary for the existence of irreducible $(p + 1)$*-tuples of matrices* M_j *satisfying (1) (proved in [Si]):*

$$\begin{array}{rcll} d_1 + \ldots + d_{p+1} & \geq & 2n^2 - 2 & \quad (\alpha_n) \\ \text{for all } j \ (r_1 + \ldots + \hat{r}_j + \ldots + r_{p+1}) & \geq & n & \quad (\beta_n) \end{array}$$

These conditions are also necessary for the existence of irreducible $(p+1)$-tuples of matrices A_j satisfying (2), see [Ko1].

Recall the basic result from [Ko1]. Its formulation depends not on the conjugacy classes C_j but only on the Jordan normal forms defined by them (as long as the eigenvalues remain generic).

For a given $(p + 1)$-tuple J^n of Jordan normal forms J_j^n (the upper index indicates the size of the matrices) define the $(p + 1)$-tuples of Jordan normal forms J^{n_ν}, $\nu = 0, \ldots, s$ as follows. Set $n_0 = n$. If J^n satisfies the condition

$$r_1 + \ldots + r_{p+1} \ \geq \ 2n \qquad (\omega_n)$$

or if it doesn't satisfy condition (β_n) or if $n=1$, then set $s=0$. If not, then set

$$n_1 = r_1 + \ldots + r_{p+1} - n \ \ \text{(hence, } n_1 < n).$$

For $j=1,\ldots,p+1$ the Jordan normal form $J_j^{n_1}$ is obtained from J_j^n *by finding (one of) the eigenvalue(s) of J_j^n with greatest number of Jordan blocks* and by decreasing by 1 the sizes of the $n-n_1$ *smallest* Jordan blocks with this eigenvalue. (Their number is $n-r_j$ which is $\geq n-n_1$ because if J^n satisfies condition (β_n), then $n_1 \geq r_j$.) Denote symbolically the construction of the Jordan normal forms $J_j^{n_1}$ by

$$\Psi : \left\{ \begin{array}{ccc} n & \mapsto & n_1 \\ (J_1^n, \ldots, J_{p+1}^n) & \mapsto & (J_1^{n_1}, \ldots, J_{p+1}^{n_1}) \end{array} \right.$$

Let the $(p+1)$-tuples of Jordan normal forms J^{n_0}, ..., $J^{n_{\nu_0}}$ be constructed. If $J^{n_{\nu_0}}$ satisfies condition $(\omega_{n_{\nu_0}})$ or if it doesn't satisfy condition $(\beta_{n_{\nu_0}})$ or if $n_{\nu_0}=1$, then set $s=n_{\nu_0}$. If not, then define $J^{n_{\nu_0+1}}$ after $J^{n_{\nu_0}}$ in the same way as J^{n_1} was defined after J^n, see Ψ. One has $n=n_0>n_1>\ldots>n_s$.

In the additive version and in the case $d=1$ of the multiplicative one the following theorem is true (see [Ko1]):

Theorem 1 *For given conjugacy classes c_j or C_j with Jordan normal forms J_j^n with generic eigenvalues there exist irreducible $(p+1)$-tuples of matrices A_j or M_j with Jordan normal forms J_j^n satisfying respectively (2) or (1) if and only if the following two conditions hold:*

i) *The $(p+1)$-tuple of Jordan normal forms J_j^n satisfies the inequalities (α_n) and (β_n);*

ii) *Either the $(p+1)$-tuple of Jordan normal forms $J_j^{n_s}$ satisfies the inequality (ω_{n_s}), or one has $n_s=1$.*

Remarks: 1) Set $d_j^\nu = d(J_j^{n_\nu})$, $d_j^0=d_j=d(J_j^n)$. Set $d_1+\ldots+d_{p+1} = 2n^2-2+\kappa$, $\kappa \in \mathbf{N}$. The number n is divisible by d and the numbers d_j are divisible by d^2. Hence, if $d>1$ and if condition (α_n) holds, then one must have $\kappa \geq 2$.

The quantity $2-\kappa$ is called *index of rigidity* (see [Ka]). Irreducible representations with $\kappa=0$ are called *rigid*; they are unique up to conjugacy, see [Ka] and [Si]. In the last remark of this section we explain how to give the exhaustive list of $(p+1)$-tuples of Jordan normal forms defining irreducible representations of a given index of rigidity and with generic eigenvalues.

2) On the other hand, one has (for all ν) $d_1^\nu+\ldots+d_{p+1}^\nu = 2(n_\nu)^2-2+\kappa$. Indeed, for all ν one has $\sum_{j=1}^{p+1} r(J_j^{n_\nu}) = n_\nu + n_{\nu+1}$ by definition. One has as well $d_j^{\nu+1} = d_j^\nu - 2(n_\nu - n_{\nu+1})r(J_j^{n_\nu})$ (to be checked directly). Hence,

$$\sum_{j=1}^{p+1} d_j^{\nu+1} = \sum_{j=1}^{p+1} d_j^\nu - 2(n_\nu - n_{\nu+1}) \sum_{j=1}^{p+1} r(J_j^{n_\nu}) =$$

$$= 2(n_\nu)^2 - 2 + \kappa - 2(n_\nu - n_{\nu+1})(n_\nu + n_{\nu+1}) = 2(n_{\nu+1})^2 - 2 + \kappa$$

Thus in the case $d>1$ one can never have $n_s=1$ because for $n_s=1$ one has $\kappa=0$.

3) It is shown in [Ko2] that if the $(p+1)$-tuple of Jordan normal forms J_j^n satisfies condition (ω_n), then one has $d_1+\ldots+d_{p+2} \geq 2n^2$. Hence, if the Jordan normal forms $J_j^{n_\nu}$ satisfy condition (ω_{n_ν}), then $d_1^\nu+\ldots+d_{p+1}^\nu \geq 2(n_\nu)^2$. This together with 1) and 2) implies that $n_s=1$ if and only if (α_n) is an equality.

Definition. Call a $(p+1)$-tuple J^n of Jordan normal forms J_j^n *good* if it satisfies conditions *i)* and *ii)* of the theorem.

Corollary 2 *Let $n > 1$. Then the $(p+1)$-tuple J^n of Jordan normal forms J_j^n is good when either it satisfies condition (ω_n) or when the $(p+1)$-tuple of Jordan normal forms $J_j^{n_1}$ is good (and only in these two cases).*

This follows from the definition of the Jordan normal forms $J_j^{n_\nu}$. Indeed, if J^n is good, then it either satisfies condition (ω_n) or the $(p+1)$-tuple J^{n_s} satisfies condition (ω_{n_s}) (recall that if $d > 1$, the possibility $n_s = 1$ is excluded). In the second case one proves by induction on $k = s - \nu$ that all $(p+1)$-tuples J^{n_ν} are good.

Theorem 1 can be reformulated as follows:

In the additive version and in the case $d = 1$ of the multiplicative one there exist for generic eigenvalues irreducible $(p+1)$-tuples of matrices A_j or M_j satisfying respectively condition (2) or (1) if and only if the $(p+1)$-tuple of Jordan normal forms J_j^n is good.

1.4 The basic results of this paper and what still remains to be done

The aim of the present paper is to prove

Theorem 3 *Conditions* i) *and* ii) *from Theorem 1 are necessary and sufficient for the existence of irreducible $(p+1)$-tuples of matrices M_j satisfying (1) and with generic eigenvalues in the case $d > 1$.*

The sufficiency of conditions *i)* and *ii)* from Theorem 1 follows from

Theorem 4 *Let $d > 1$ and let ξ be a primitive root of unity. Then conditions* i) *and* ii) *from Theorem 1 are sufficient for the existence of $(p+1)$-tuples of matrices M_j with trivial centralizers.*

Theorem 4 is stronger than the proof of the sufficiency in Theorem 3 because the eigenvalues are not presumed generic.

Theorem 5 *If $d > 1$ and if $d_1 + \ldots + d_{p+1} \geq 2n^2 + 2$, then conditions* i) *and* ii) *from Theorem 1 are necessary and sufficient for the existence of $(p+1)$-tuples of matrices M_j satisfying (1) and with trivial centralizers. If the eigenvalues are relatively generic, then there exist irreducible such $(p+1)$-tuples.*

Notice that the theorem does not require primitivity of ξ. The last two theorems are proved in Section 5.

Lemma 6 *A) If $q > 1$, if $d_1 + \ldots + d_{p+1} = 2n^2$, if all $(p+1)$ Jordan normal forms are diagonal, if the eigenvalues $\sigma_{k,j}$ are relatively generic and if ξ is not primitive, then such a $(p+1)$-tuple of matrices M_j satisfying (1) (if it exists) is with trivial centralizer if and only if it is irreducible.*

B) Let $d_1 + \ldots + d_{p+1} \geq 2n^2 + 2$ and let for the rest the conditions from A) hold. Then if there exist $(p+1)$-tuples of matrices M_j satisfying (1), then there exist irreducible ones as well.

The lemma is proved in Subsection 3.2.

Theorem 7 *Conditions* i) *and* ii) *from Theorem 1 are necessary for the existence of $(p+1)$-tuples of matrices M_j or A_j satisfying (1) or (2) and with a trivial centralizer.*

The theorem claims necessity in all possible situations. It is proved in Section 4.

For generic eigenvalues Problem (I) (and Problem (TC) as well) is completely solved by Theorems 1 and 3.

For non-generic eigenvalues we focus on Problem (TC). The situations in which the answer is known are given by Theorems 4 and 5. It is shown in [Ko1] that conditions *i)* and *ii)* from Theorem 1 are necessary and sufficient for the solvability of Problem (TC) when $d = 1$, (α_n) being a strict inequality.

The cases in which Problem (TC) remains to be considered for non-generic eigenvalues (in both versions – additive or multiplicative) and the conjectures the author makes are:

1) (α_n) is an equality (this implies $d = 1$, see the remarks after Theorem 1).

Conjecture. *Conditions* i) *and* ii) *from Theorem 1 are necessary and sufficient if $q = 1$. For $q > 1$ there are cases in which they are and cases in which they are not sufficient.*

2) $d > 1$, $d_1 + \ldots + d_{p+1} = 2n^2$ and in the multiplicative version ξ is not primitive.

Conjecture. *There exist no $(p+1)$-tuples of matrices A_j or M_j, satisfying (2) or (1), with trivial centralizers.*

Remark: It is possible to give an exhaustive list of the $(p+1)$-tuples of Jordan normal forms of a given size n admitting generic eigenvalues and defining irreducible representations of a fixed index of rigidity. Explain it first for rigid representations (for them one has $n_s = 1$), with diagonal Jordan normal forms J_j^n.

Denote by $\Xi(m)$ the set of all $(p+1)$-tuples of diagonal Jordan normal forms satisfying conditions *i)* and *ii)* from Theorem 1, of size $\leq m$ (scalar Jordan normal forms are allowed). Find then all $(p+1)$-tuples of diagonal Jordan normal forms not from $\Xi(m)$ which after applying the map Ψ result in a $(p+1)$-tuple from $\Xi(m)$. To this end one has to assume that every diagonal Jordan normal form of a $(p+1)$-tuple from $\Xi(m)$ has an eigenvalue of multiplicity 0 (which eventually was of positive multiplicity before applying Ψ). Thus one can obtain $\Xi(m+1)$ (one will have to exclude the $(p+1)$-tuples of size $> m+1$).

Having obtained the set $\Xi(n)$, we leave only the $(p+1)$-tuples of size exactly n. Denote their set by Θ. Explain how to obtain the analog of Θ obtained when the requirement the Jordan normal forms to be diagonal to be dropped. One constructs the set of $(p+1)$-tuples of Jordan normal forms $J_j^{n,1}$ such that the $(p+1)$-tuple of corresponding diagonal Jordan normal forms belongs to Θ – this defines the set Φ. After this one excludes from Φ the $(p+1)$-tuples not admitting generic eigenvalues – this gives the necessary list for index of rigidity equal to 2.

For index of rigidity $h \leq 0$ one has to give first the list L_h of $(p+1)$-tuples of diagonal Jordan normal forms satisfying condition (ω_n); after this one constructs the sets $\Xi(n)$, Θ and Φ by analogy with the case of index of rigidity equal to 2.

For index of rigidity $h = 0$ the list L_0 contains only three triples and one quadruple of diagonal Jordan normal forms, with MVs equal to a) (1,1), (1,1), (1,1), (1,1) b) (1,1,1), (1,1,1), (1,1,1) c) (1,1,1,1), (1,1,1,1), (2,2) d) (1,1,1,1,1,1), (2,2,2), (3,3). The reader will be able to deduce this fact from the beginning of the last section and from Lemma 3 from [Ko2].

If one wants to obtain the list L_h for $h \leq -2$, then one has to assume that L_0 contains four series of diagonal Jordan normal forms. They are a) (d,d), (d,d), (d,d), (d,d) b) (d,d,d), (d,d,d), (d,d,d) c) (d,d,d,d), (d,d,d,d), (2d,2d) d) (d,d,d,d,d,d), (2d,2d,2d), (3d,3d) where $d \in \mathbf{N}$ is such that the size of the matrices be $\leq n$.

For negative indices of rigidity the list L_h can be given by using again the ideas of the last section. First of all, we operate instad of with diagonal Jordan normal forms with their corresponding Jordan normal forms with a single eigenvalue. Having found the list L_{h-2} one obtains the list L_h by using *operations* (s, l) (defined in **2** of Subsubsection 6.3.1) and the inverse of the *merging* (defined in **6** of that subsubsection. The details are left for the reader.

2 Definitions and notation

2.1 Levelt's result and its corollaries

In [L] Levelt describes the structure of the solution to a regular system at a pole:

Theorem 8 *In a neighbourhood of a pole the solution to the regular linear system*

$$\dot{X} = A(t)X \tag{3}$$

can be represented in the form

$$X = U_j(t-a_j)(t-a_j)^{D_j}(t-a_j)^{E_j}G_j \tag{4}$$

where U_j is holomorphic in a neighbourhood of the pole a_j, $D_j = diag\,(\varphi_{1,j}, \ldots, \varphi_{n,j})$, $\varphi_{n,j} \in \mathbf{Z}$, $G_j \in GL(n, \mathbf{C})$. The matrix E_j is in upper-triangular form and the real parts of its eigenvalues belong to $[0,1)$ (by definition, $(t-a_j)^{E_j} = e^{E_j \ln(t-a_j)}$). The numbers $\varphi_{k,j}$ satisfy the condition (6) formulated below.

System (3) is fuchsian at a_j if and only if

$$\det U_j(0) \neq 0 \tag{5}$$

We formulate the condition on $\varphi_{k,j}$. Let E_j have one and the same eigenvalue in the rows with indices $s_1 < s_2 < \ldots < s_q$. Then we have

$$\varphi_{s_1,j} \geq \varphi_{s_2,j} \geq \ldots \geq \varphi_{s_q,j} \tag{6}$$

Remarks: 1) Denote by $\beta_{k,j}$ the diagonal entries (i.e. the eigenvalues) of the matrix E_j. If the system is fuchsian, then the sums $\beta_{k,j} + \varphi_{k,j}$ are the eigenvalues $\lambda_{k,j}$ of the matrix-residuum A_j, see [Bo1], Corollary 2.1.

2) The numbers $\varphi_{k,j}$ are defined as valuations in the solution eigensubspace for the eigenvalue $\exp(2\pi i \beta_{k,j})$ of the monodromy operator, see the details in [L]. These valuations can be defined on each subspace invariant for the monodromy operator.

3) One can assume without loss of generality that equal eigenvalues of E_j occupy consecutive positions on the diagonal and that the matrix E_j is block-diagonal, with diagonal blocks of sizes equal to their multiplicities. The diagonal blocks themselves are upper-triangular.

4) An improvement of Levelt's form (4) can be found in [Ko3]. More precisely, the fact that all entries of E_j above the diagonal can be made equal to 0 or to 1.

5) Let $q > 1$. The rest of the division of $\sum_{j=1}^{p+1} \sum_{k=1}^{n} \beta_{k,j}$ (which is an integer) by q equals m_0.

6) In [Bo1] A.Bolibrukh proves the following lemma (see Lemma 3.6 there):

Lemma 9 *The sum of the numbers $\varphi_{k,j} + \beta_{k,j}$ of system (3) corresponding to a subspace of the solution space invariant for all monodromy operators is a non-positive integer.*

Corollary 10 *Let the sum of the eigenvalues $\lambda_{k,j}$ of the matrices-residua of a fuchsian system corresponding to a subspace of the solution space of dimension m invariant for all monodromy operators be 0. Denote these eigenvalues by $\lambda'_{k,j}$. Then there exists a change $X \mapsto RX$, $R \in GL(n, \mathbf{C})$ after which the system becomes block upper-triangular, the left upper block being of size m, the restrictions of the matrices-residua to it having eigenvalues $\lambda'_{k,j}$.*

A proof of the corollary can be found in [Bo2].

2.2 Subordinate Jordan normal forms, canonical and strongly generic eigenvalues

Definition. Given two conjugacy classes c', c'' with one and the same eigenvalues, of one and the same multiplicities, we say that c'' is *subordinate* to c' if c'' lies in the closure of c', i.e. for any matrix $A \in c''$ there exists a deformation $\tilde{A}(\varepsilon)$, $\tilde{A}(0) = A$ such that for $0 \neq \varepsilon \in (\mathbf{C}, 0)$ one has $\tilde{A}(\varepsilon) \in c'$. Given two Jordan normal forms J', J'', we say that J'' is *subordinate* to J' if there exist conjugacy classes $c' \in J'$, $c'' \in J''$ such that c'' is subordinate to c'.

In the text we denote by A_j matrices from $gl(n, \mathbf{C})$. They are often interpreted as matrices-residua of fuchsian systems on Riemann's sphere, i.e. systems of the form

$$\dot{X} = (\sum_{j=1}^{p+2} A_j/(t - a_j))X \qquad (7)$$

If this system has no pole at infinity, then one has $A_1 + \ldots + A_{p+2} = 0$.

Remark. System (7) has $p + 2$ poles because in what follows we have to realize the monodromy groups by fuchsian systems having an additional singularity with scalar local monodromy. When $A_{p+2} = 0$ the system has $p + 1$ poles.

Definition. The eigenvalues $\lambda_{k,j}$ of the matrix-residuum A_j are called *canonical* if none of the differences between two of its eigenvalues is a non-zero integer. The eigenvalues of the corresponding *monodromy operators* M_j equal $\exp(2\pi i \lambda_{k,j})$. Hence, if the eigenvalues of A_j are canonical, then to equal (to different) eigenvalues of the corresponding monodromy operator there correspond equal (different) eigenvalues of A_j.

Proposition 11 *If the eigenvalues of the matrix A_j are canonical, then one has $J(A_j) = J(M_j)$ and M_j is conjugate to* $\exp(2\pi i A_j)$.

The proof can be found in [Wa].

Definition. Eigenvalues $\lambda_{k,j}$ satisfying none of the equalities (γ) modulo $\mathbf{Z}$ (see the previous section) are called *strongly generic*; this definition is given only for eigenvalues $\lambda_{k,j}$; they are strongly generic if and only if the corresponding eigenvalues $\sigma_{k,j} = \exp(2\pi i \lambda_{k,j})$ are generic.

Definition. If $q > 1$ and if the eigenvalues $\lambda_{k,j}$ satisfy none of the equalities (γ) modulo $\mathbf{Z}$ except (γ^*) and its multiples (see Subsection 1.2), then the eigenvalues are called *strongly relatively generic*.

Definition. Let the eigenvalues $\lambda_{k,j}$ not be strongly generic. Hence, there holds at least one equality of the form

$$\sum_{j=1}^{p+2} \sum_{k \in \Phi_j} \lambda_{k,j} = m \ , \ m \in \mathbf{Z} \ , \ \sharp\Phi_1 = \ldots = \sharp\Phi_{p+2} = \kappa \ , \ 1 \leq \kappa < n \qquad (\gamma')$$

The number $|m|$ is called *distance of the set of eigenvalues to the relation* (γ'). The minimal of the numbers $|m|$ (over all relations (γ')) is called *distance of the set of eigenvalues to the set of non-generic eigenvalues* (or just *distance*). For generic eigenvalues their distance is by definition equal to ∞.

Lemma 12 **A)** *Let the eigenvalues $\sigma_{k,j}$ defined by the conjugacy classes C_j be non-generic and either with a simple PMV or with a non-simple one ξ being a primitive root of unity, and let at least one of the classes C_j (say, C_1) have at least two different eigenvalues. Then for every $h \in \mathbf{N}$ sufficiently large there exist eigenvalues $\lambda_{k,j}$ with zero sum such that*

1) for all k, j one has $\exp(2\pi i\lambda_{k,j}) = \sigma_{k,j}$;

2) for $j \leq p$ the eigenvalues $\lambda_{k,j}$ are canonical;

3) for $j = p+1$ if $\lambda_{k_1,j} - \lambda_{k_2,j} \in \mathbf{Z}$, then $\lambda_{k_1,j} - \lambda_{k_2,j} = 0$ or ± 1;

4) the distance of the eigenvalues is $\geq h$.

B) *If ξ is not primitive (the rest of the conditions being like in A)), then there exist eigenvalues $\lambda_{k,j}$ satisfying conditions 1), 2), 3) and*

4') for every relation (γ') which is not a multiple of (γ^*) its distance is $\geq h$.

Before proving the lemma we deduce from it

Corollary 13 *If the eigenvalues $\lambda_{k,j}$ of the matrices-residua A_j of system (7) are like in the lemma, with $h > n$, and if the $(p+1)$-tuple of matrices A_j is irreducible, then the monodromy group of the system is with trivial centralizer.*

Notice that the irreducibility of the $(p+1)$-tuple of matrices A_j is not automatic when ξ is not primitive because there holds (γ^*).

Proof:

1^0. Suppose first (see $1^0 - 3^0$) that ξ is primitive. Assume that the monodromy group of system (7) satisfying the conditions of the corollary is with non-trivial centralizer $\mathcal{Z}$. Then $\mathcal{Z}$ either

a) contains a diagonalizable matrix D with exactly two distinct eigenvalues or

b) contains a nilpotent matrix N with $N^2 = 0$.

(Indeed, if $X \in \mathcal{Z}$, then every polynomial of the semi-simple or of the nilpotent part of X belongs to $\mathcal{Z}$ which allows to construct such matrices D or N.)

2^0. In case a) one can assume that D is diagonal – $D = \begin{pmatrix} \alpha I & 0 \\ 0 & \beta I \end{pmatrix}$. Hence, the monodromy operators are of the form $M_j = \begin{pmatrix} M_j' & 0 \\ 0 & M_j'' \end{pmatrix}$. Applying Lemma 9 to the sums λ', λ'' of the eigenvalues $\lambda_{k,j}$ corresponding respectively to the $(p+1)$-tuples of blocks M_j', M_j'', one obtains the inequalities $\lambda' \leq 0$, $\lambda'' \leq 0$. On the other hand, $\lambda' + \lambda'' = 0$. Hence, $\lambda' = \lambda'' = 0$ which contradicts the condition $h > 0$.

3^0. In case b) one can assume that $N = \begin{pmatrix} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $M_j = \begin{pmatrix} M_j' & * & * \\ 0 & M_j'' & * \\ 0 & 0 & M_j' \end{pmatrix}$ (the form of N can be achieved by conjugation, the one of M_j results from $[M_j, N] = 0$).

Denote by λ^1, λ^2, λ^3 the sums of the eigenvalues $\lambda_{k,j}$ corresponding respectively to the $(p+1)$-tuples of upper blocks M_j', of blocks M_j'', of lower blocks M_j'. One has $\lambda^1 + \lambda^2 + \lambda^3 = 0$, $\lambda^1 \leq 0$, $\lambda^1 + \lambda^2 \leq 0$, $\lambda^2 + \lambda^3 \geq 0$ $\lambda^3 \geq 0$ (Lemma 9) and $|\lambda^1 - \lambda^3| < n$ because the eigenvalues $\lambda_{k,j}$ are canonical for $j \leq p$ and for $j = p+1$ there holds condition 3) of the lemma (and the blocks M_j', M_j'' are of sizes $< n$). Hence, either $|\lambda^1 + \lambda^2| < n$ or $|\lambda^2 + \lambda^3| < n$ which is a contradiction with the distance of the set of eigenvalues to the set of non-generic eigenvalues being $> n$.

4^0. Suppose now that ξ is not primitive. If the monodromy group of the system is with non-trivial centralizer, then in case a) one obtains $\lambda' = \lambda'' = 0$. This is still not a contradiction with the condition $h > 0$ because there remains the relation (γ^*).

By Corollary 10, one can make a change $X \mapsto RX$ and block-triangularize the matrices-residua of the system. The change results in $A_j \mapsto R^{-1}A_jR$. This contradicts the condition the $(p+1)$-tuple of matrices A_j to be irreducible.

In case b) one has $\lambda^1 + \lambda^2 + \lambda^3 = 0$, $\lambda^1 \leq 0$, $\lambda^1 + \lambda^2 \leq 0$, $\lambda^2 + \lambda^3 \geq 0$ $\lambda^3 \geq 0$ (Lemma 9) and $|\lambda^1 - \lambda^3| < n$ (like in 3^0). This is possible only if $\lambda^1 = \lambda^2 = \lambda^3 = 0$ and each of the last

equalities results from (γ^*). By Corollary 10, one can block-triangularize the matrices-residua of the system by a change $X \mapsto RX$. This again contradicts the irreducibility of the $(p+1)$-tuple of matrices A_j.

The corollary is proved.

Proof of Lemma 12:

1^0. Prove A) (see 1^0 – 7^0). Fix some canonical eigenvalues $\lambda^0_{k,j}$ satisfying condition 1) whose sum eventually is non-zero. Denote by l the rest of the division of their sum (which is necessarily integer) by n. Decrease l of the eigenvalues $\lambda^0_{k,p+1}$ by 1 (and don't change the other eigenvalues $\lambda_{k,j}$) – this defines the eigenvalues $\lambda^1_{k,j}$.

2^0. Fix integers $g_{k,j}$ such that

a) to equal eigenvalues $\lambda^0_{k,j}$ there correspond equal integers $g_{k,j}$ and

b) $\sum_{k=1}^{n}\sum_{j=1}^{p+1}(\lambda^1_{k,j} + g_{k,j}) = 0$.

Set $\lambda^2_{k,j} = \lambda^1_{k,j} + g_{k,j}$.

3^0. Denote by λ'_j, λ''_j two different eigenvalues $\lambda^0_{k,j}$, of multiplicities m', m''. Change their corresponding numbers $g_{k,j}$ (denoted by g'_j, g''_j) respectively to $g'_j + um''$, $g''_j - um'$, $u \in \mathbf{N}$. This defines a new set of eigenvalues $\lambda^2_{k,j}$. (The other integers $g_{k,j}$ do not change.)

4^0. For a given relation (γ') (satisfied by the eigenvalues $\lambda^0_{k,j}$ instead of $\lambda_{k,j}$) call *quasi-multiplicity* of an eigenvalue $\lambda^0_{k,j}$ the number of eigenvalues $\lambda^0_{i,j}$ equal to $\lambda^0_{k,j}$ such that $i \in \Phi_j$ (the sets Φ_j were defined in Subsection 1.1). For each given relation (γ') the quasi-multiplicities of the eigenvalues $\lambda^0_{k,j}$ are not proportional to their multiplicities because either the PMV is simple or it is not but ξ is a primitive root of unity.

5^0. For each relation (γ') and for each couple of eigenvalues λ'_j, λ''_j like in 3^0 either

1') for all values of $u \in \mathbf{N}$ the distance of the set of eigenvalues to the relation (γ') remains the same or

2') only for finitely many of them this distance is $< h$.

6^0. For a given relation (γ') there exists a couple of eigenvalues λ'_j, λ''_j like in 3^0 for which there holds 2'). Having chosen this couple, denote by Ξ the set of relations (γ') for which the chosen couple satisfies 2'). Hence, for the chosen couple one can find a finite subset N_0 of $\mathbf{N}$ such that for $u \in (\mathbf{N}\backslash N_0)$ the distance of the set of eigenvalues $\lambda^2_{k,j}$ to each of the relations from Ξ is $\geq h$. Fix $u = u_0 \in (\mathbf{N}\backslash N_0)$. This means that we change the eigenvalues $\lambda^2_{k,j}$.

7^0. If Ξ is not the set of all relations (γ'), then choose such a relation not from it and repeat what was done in 6^0. Every time we do this, the distance of the set of eigenvalues to more and more relations (γ') becomes $\geq h$ and the distance to the rest of them does not change. As there are finitely many relations (γ'), after finitely many steps the distance of the set of eigenvalues $\lambda^2_{k,j}$ to the set of non-generic eigenvalues becomes $\geq h$, i.e. condition 4) holds. Conditions 1) – 3) hold by construction.

8^0. The proof of B) is completely analogous. One first divides the multiplicities of all eigenvalues by (m_0, q), then constructs the eigenvalues $\lambda_{k,j}$ like in case A) and then one multiplies the multiplicities by (m_0, q). The distances of (γ^*) and of its multiples remain 0, the distances of the other relations are $\geq (m_0, q)h > h$.

The lemma is proved.

2.3 Corresponding Jordan normal forms and normalized chains

In [Ko1] we define *correspondence* between Jordan normal forms. Namely, for each Jordan normal form J_0 we define its *corresponding* diagonal Jordan normal form J_1 as follows.

Suppose first that J_0 has a single eigenvalue of multiplicity n. Replace each Jordan block of J_0 of size $q' \times q'$ by a diagonal $q' \times q'$-matrix with q' distinct eigenvalues where the last eigenvalues of the diagonal matrices replacing Jordan blocks of J_0 are the same, their last but one eigenvalues are the same and different from the last ones and for all $s \geq 0$ their last but s eigenvalues (denoted by h_s) are the same and different from the last but l ones for $l < s$.

If J_0 has several eigenvalues, then this procedure is performed for every eigenvalue with the requirement eigenvalues of J_1 corresponding to different eigenvalues of J_0 to be different.

Remark: The definition of J_1 after J_0 can be described in equivalent terms like this. Call *Jordan normal form (JNF) of size* n a family $J^n = \{b_{i,l}\}$ ($i \in I_l$, $I_l = \{1, \ldots, s_l\}$, $l \in L$) of positive integers $b_{i,l}$ whose sum is n. Here L is the set of eigenvalues (all distinct) and I_l is the set of Jordan blocks with eigenvalue l, $b_{i,l}$ is the size of the i-th block with this eigenvalue. An $n \times n$-matrix Y has the JNF J^n (notation: $J(Y) = J^n$) if to its distinct eigenvalues λ_l, $l \in L$, there belong Jordan blocks of sizes $b_{i,l}$. For a given JNF $J^n = \{b_{i,l}\}$ define its *corresponding* diagonal JNF J'^n. A diagonal JNF is a partition of n defined by the multiplicities of the eigenvalues. For each l $\{b_{i,l}\}$ is a partition of $\sum_{i \in I_l} b_{i,l}$ and J' is the disjoint sum of the dual partitions. Thus if for each fixed l one has $b_{1,l} \geq \ldots \geq b_{s_l,l}$, then each eigenvalue $l \in L$ is replaced by $b_{1,l}$ new eigenvalues $h_{1,l}$, $\ldots$, $h_{b_{1,l},l}$ (hence, J'^n has $\sum_{l \in L} b_{1,l}$ distinct eigenvalues). For l fixed, set g_k for the multiplicity of the eigenvalue $h_{k,l}$. Then the first $b_{s_l,l}$ numbers g_k equal s_l, the next $b_{s_l-1,l} - b_{s_l,l}$ equal $s_l - 1$, $\ldots$, the last $b_{1,l} - b_{2,l}$ equal 1.

Definition. Two Jordan normal forms *correspond* to one another if they correspond to one and the same diagonal Jordan normal form (and thus the correspondence of Jordan normal forms is a relation of equivalence).

Definition. We say that the eigenvalues of a Jordan matrix G with Jordan normal form J_1 form a *normalized chain w.r.t. the Jordan normal form* J_0 if for every eigenvalue of J_0 and for all s one has $h_s - h_{s-1} \in \mathbf{N}^+$ and any two eigenvalues of G corresponding to different eigenvalues of J_0 are non-congruent modulo $\mathbf{Z}$.

The following properties of corresponding Jordan normal forms are proved in [Ko1]:

Theorem 14 *1) If the Jordan normal forms J_0 and J_1 correspond to one another, then $r(J_0) = r(J_1)$ and $d(J_0) = d(J_1)$.*

2) To each Jordan normal form $J' = \{b'_{i,l}\}$ there corresponds a single Jordan normal form with a single eigenvalue $J'' = \{b''_{k,1}\}$. One has $b''_{k,1} = \sum_l b'_{k,l}$, $k = 1, \ldots, s_1$.

3) Let the two Jordan normal forms $J^{n\prime}$ and $J^{n\prime\prime}$ correspond to one another. Choose in each of them an eigenvalue with maximal number of Jordan blocks. By 1) these numbers coincide. Denote them by k'. Decrease by 1 the sizes of the k smallest Jordan blocks with these eigenvalues where $k \leq k'$. Then the two Jordan normal forms of size $n - k$ obtained in this way correspond to one another.

4) Denote by G^0, G^1 two Jordan matrices with Jordan normal forms J_0, J_1 corresponding to each other, where G^1 is diagonal and G^0 is nilpotent and they are defined like J_1 and J_0 from the beginning of the subsection (when J_1 and J_0 are considered like matrices (J_0 being Jordan), not like Jordan normal forms). Then the orbits of the matrices εG^1 and $G^0 + \varepsilon G^1$ are the same for $\varepsilon \in \mathbf{C}^*$.

5) If G^0 is not necessarily nilpotent and if G^0, G^1 are defined by analogy with J_0, J_1, then the matrix $G^0 + \varepsilon G^1$ is diagonalizable and its Jordan normal form is J_1 if $\varepsilon \in \mathbf{C}^*$ is small enough.

Corollary 15 *Consider two $(p+1)$-tuples of Jordan normal forms – J_j^n and $J_j^{n\prime}$ – where for all j the two Jordan normal forms J_j^n and $J_j^{n\prime}$ correspond to one another. Construct for each*

of the two $(p+1)$-tuples the $(p+1)$-tuples of Jordan normal forms $J_j^{n_\nu}$ and $J_j^{n_\nu{}'}$ like explained before Theorem 1. Then one has $n_\nu = n_{\nu'}$ for all ν, and for all ν and all j the Jordan normal forms $J_j^{n_\nu}$ and $J_j^{n_\nu{}'}$ correspond to one another.

The corollary follows from 1) and 3) of the above theorem.

Denote by J and J' an arbitrary Jordan normal form and its corresponding Jordan normal form with a single eigenvalue. Consider a couple D, D' of Jordan matrices with these Jordan normal forms. Suppose that the Jordan blocks of D of sizes $b_{k,l}$ for k fixed are situated in the same rows where is situated the Jordan block of size $\sum_l b_{k,l}$ of D' (see 2) of the above theorem). Denote by D_0 the diagonal (i.e. semi-simple) part of the matrix D.

Proposition 16 *For all $\varepsilon \neq 0$ the matrix $\varepsilon D_0 + D'$ is conjugate to εD.*

Remark: Permute the diagonal entries of D_0 so that before and after the permutation each entry remains in one of the rows of one and the same Jordan block of D'. Then the proposition holds again and the proof is the same, see Proposition 20 from [Ko1].

Notation. We denote by a_j the poles of the fuchsian system (7) and by α_j the quantities $1/(a_j - a_{p+2})$. We often use the matrix $\mathcal{H}$ =diag$(-1,\ldots,-1,0,\ldots,0)$ (m_0 times -1 and $n-m_0$ times 0). The matrix with a single entry (equal to 1) in position (i,j) is denoted by $E_{i,j}$. In all other cases double subscripts denote matrix entries.

2.4 Plan of the paper

The main difficulty in the case $d>1$ is the impossibility to realize the monodromy groups by fuchsian systems with canonical eigenvalues. If the eigenvalues $\lambda_{k,j}$ are represented in the form $\beta_{k,j}+\varphi_{k,j}$ with $\varphi_{k,j}\in\mathbf{Z}$ and with $\mathrm{Re}(\beta_{k,j})\in[0,1)$, then the sum $\sum_{j=1}^{p+1}\beta_{k,j}$ is an integer (the sum of all eigenvalues $\lambda_{k,j}$ is 0, see (2)). If this integer is not divisible by d, then there exists no set of canonical eigenvalues $\lambda_{k,j}$ (canonical for each index j).

Therefore we realize such monodromy groups by fuchsian systems having $(p+2)$-nd singularities with scalar local monodromy. More exactly – equal to id (such singularities are called *apparent*. (This is why in the formulation of the problem we speak about $(p+1)$ matrices M_j and here we consider $(p+2)$-tuples of matrices A_j.) The eigenvalues at the first $(p+1)$ singular points are canonical. The ones at the $(p+2)$-nd singularity compensate the rest m_0 of the division by d of the above sum ($1\leq m_0\leq d-1$).

The necessity to operate with systems with additional singularities explains why we begin the present second part of [Ko1] by the study of $(p+2)$-tuples of matrices $A_j\in gl(n,\mathbf{C})$ with zero sum and satisfying certain linear equalities (see Subsection 3.3). For a fuchsian system defined by such a $(p+2)$-tuple of matrices for fixed poles these equalities provide the necessary and sufficient condition the monodromy around one of the singularities of the fuchsian system (7) to be scalar.

Section 3 contains also the modification of the *basic technical tool* used in [Ko1]. It is used to construct analytic deformations of $(p+2)$-tuples of matrices A_j with trivial centralizer and satisfying certain linear equalities.

Section 4 contains the proof of the necessity of conditions *i)* and *ii)* for the existence of irreducible $(p+1)$-tuples of matrices M_j satisfying (1) in the case $d>1$; Section 5 contains the proof of their sufficiency. In the proof of the sufficiency we use some results concerning the existence of irreducible $(p+1)$-tuples of nilpotent matrices with zero sum, see Section 6.

3 Some preliminaries

3.1 The basic technical tool

The *basic technical tool* is a way to deform a $(p+1)$-tuple of matrices M_j or A_j with a trivial centralizer into one with either new eigenvalues and the same Jordan normal forms of the respective matrices, or into one in which some of these Jordan normal forms J_j^n are replaced by other Jordan normal forms $J_j^{n\prime}$ where J_j^n is subordinate to $J_j^{n\prime}$, or where $J_j^{n\prime}$ corresponds to J_j^n. Explain how it works in the multiplicative version first.

Given a $(p+1)$-tuple of matrices M_j^1 satisfying condition (1) and whose centralizer is trivial, look for M_j of the form

$$M_j = (I+\varepsilon X_j(\varepsilon))^{-1}(M_j^1+\varepsilon N_j(\varepsilon))(I+\varepsilon X_j(\varepsilon))$$

where the given matrices N_j depend analytically on $\varepsilon \in (\mathbf{C},0)$ and one looks for X_j analytic in ε. (One can set $M_j^1 = Q_j^{-1}G_jQ_j$, $N_j = Q_j^{-1}V_j(\varepsilon)Q_j$ where G_j are Jordan matrices and $Q_j \in GL(n,\mathbf{C})$.)

The matrices M_j must satisfy equality (1). In first approximation w.r.t. ε this yields

$$\sum_{j=1}^{p+1} M_1^1 \dots M_{j-1}^1([M_j^1, X_j(0)] + N_j(0))M_{j+1}^1 \dots M_{p+1}^1 = 0$$

or

$$\sum_{j=1}^{p+1} P_{j-1}([M_j^1, X_j(0)(M_j^1)^{-1}] + N_j(0)(M_j^1)^{-1})P_{j-1}^{-1} = 0 \tag{8}$$

with $P_j = M_1^1 \dots M_j^1$, $P_{-1} = I$ (recall that there holds (1), therefore $M_j^1M_{j+1}^1 \dots M_{p+1}^1 = P_{j-1}^{-1}$).

Condition (1) implies that $\det M_1 \dots \det M_{p+1} = 1$. There holds $\det M_j = \det M_j^1 \det(I + \varepsilon(M_j^1)^{-1}N_j) = (\det M_j^1)(1+\varepsilon\mathrm{tr}((M_j^1)^{-1}N_j(0)) + o(\varepsilon))$. As $\det M_1^1 \dots \det M_{p+1}^1 = 1$, one has $\mathrm{tr}(\sum_{j=1}^{p+1}(M_j^1)^{-1}N_j(0))= 0$ (term of first order w.r.t. ε in $\det M_1 \dots \det M_{p+1}$).

Equation (8) admits the following equivalent form:

$$\sum_{j=1}^{p+1}([S_j, Z_j] + T_j) = 0 \tag{9}$$

with $S_j = P_{j-1}M_j^1P_{j-1}^{-1}$, $Z_j = P_{j-1}X_j(0)(M_j^1)^{-1}P_{j-1}^{-1}$, $T_j = P_{j-1}N_j(0)(M_j^1)^{-1}P_{j-1}^{-1}$.

The centralizers of the $(p+1)$-tuples of matrices M_j^1 and S_j coincide (to be checked directly), i.e. they are both trivial. There holds

Proposition 17 *The $(p+1)$-tuple of matrices A_j is with trivial centralizer if and only if the mapping $(sl(n,\mathbf{C}))^{p+1} \to sl(n,\mathbf{C})$, $(X_1,\dots,X_{p+1}) \mapsto \sum_{j=1}^{p+1}[A_j, X_j]$ is surjective.*

Proof: The mapping is not surjective if and only if the images of all mappings $X_j \mapsto [A_j, X_j]$ belong to one and the same proper linear subspace of $sl(n,\mathbf{C})$ which can be defined by a condition of the form $\mathrm{tr}(D[A_j, X_j]) = 0$ for all $X_j \in sl(n,\mathbf{C})$ where $0 \neq D \in sl(n,\mathbf{C})$. This amounts to $\mathrm{tr}([D, A_j]X_j) = 0$ for all $X_j \in sl(n,\mathbf{C})$, i.e. $[D, A_j] = 0$ for $j = 1,\dots,p+1$.

The proposition is proved.

The mapping

$$(sl(n,\mathbf{C}))^{p+1} \to sl(n,\mathbf{C}) \ , \ (Z_1,\ldots,Z_{p+1}) \mapsto \sum_{j=1}^{p+1}[S_j,Z_j]$$

is surjective (Proposition 17). Recall that $\mathrm{tr}(\sum_{j=1}^{p+1}(M_j^1)^{-1}N_j(0)) = 0$, i.e. $\mathrm{tr}(\sum_{j=1}^{p+1}T_j) = 0$. Hence, equation (9) is solvable w.r.t. the unknown matrices Z_j and, hence, equation (8) is solvable w.r.t. the matrices $X_j(0)$. The implicit function theorem implies (we use the surjectivity here) that one can find X_j analytic in $\varepsilon \in (\mathbf{C},0)$, i.e. one can find the necessary matrices M_j.

In the additive version one has matrices $A_j^1 = Q_j^{-1}G_jQ_j$ instead of M_j^1 and one sets

$$\tilde{A}_j = (I+\varepsilon X_j(\varepsilon))^{-1}Q_j^{-1}(G_j+\varepsilon V_j(\varepsilon))Q_j(I+\varepsilon X_j(\varepsilon))$$

where $V_j(\varepsilon)$ are given matrices analytic in ε; one has $\mathrm{tr}(\sum_{j=1}^{p+1}V_j(\varepsilon)) \equiv 0$. The matrices $X_j(0)$ satisfy the equation (which is of the form (9))

$$\sum_{j=1}^{p+1}[A_j^1, X_j(0)] = -\sum_{j=1}^{p+1}Q_j^{-1}V_jQ_j \ .$$

The existence of X_j analytic in ε is justified like in the multiplicative version.

Lemma 18 *If for given Jordan normal forms J_j^n of the matrices M_j and generic eigenvalues there exists irreducible $(p+1)$-tuples of such matrices (satisfying (1)), then there exist such $(p+1)$-tuples for M_j from the corresponding diagonal Jordan normal forms (and with generic eigenvalues).*

Proof: 1^0. Denote by J_j' and J_j'' an arbitrary and its corresponding diagonal Jordan normal form. Let the $(p+1)$-tuple of matrices M_j^0 be irreducible, with generic eigenvalues and satisfying (1). Set $M_j^0 = Q_j^{-1}G_jQ_j$ where G_j are Jordan matrices and $J(G_j) = J_j'$.

2^0. Construct a deformation of the $(p+1)$-tuple of the form

$$M_j = (I+\varepsilon X_j(\varepsilon))^{-1}Q_j^{-1}(G_j+\varepsilon L_j)Q_j(I+\varepsilon X_j(\varepsilon))$$

where X_j are analytic in $\varepsilon \in (\mathbf{C},0)$ and L_j are diagonal matrices with $J(L_j) = J_j''$. More exactly, assume that G_j and L_j are defined respectively like G^0 and G^1 from 4) of Theorem 14. By 5) of that theorem, one has $J(M_j) = J(L_j)$ for $\varepsilon \neq 0$ small enough.

3^0. The basic technical tool provides the existence of irreducible $(p+1)$-tuples of matrices M_j for $\varepsilon \neq 0$ small enough. Their Jordan normal forms are J_j'' and their eigenvalues are generic.

The lemma is proved.

3.2 Proof of Lemma 6

1^0. Prove A) (see $1^0 - 5^0$). Suppose that there exists such a $(p+1)$-tuple of matrices M_j with trivial centralizer but not irreducible. Then it can be conjugated to a block upper-triangular form with irreducible diagonal blocks whose eigenvalue satisfy the only non-genericity relation (γ^*) (defined in Subsection 1.2) and eventually some of its multiples.

2^0. Consider two diagonal blocks and the representations Φ_1, Φ_2 (of sizes m_1, m_2) which they define. The Jordan normal forms of each of the matrices M_j restricted to each of these blocks is a multiple of one and the same diagonal Jordan normal form and the ratio of the

multiplicities of one and the same eigenvalues of M_j as eigenvalues of Φ_1 and Φ_2 equals m_1/m_2 for every eigenvalue.

3^0. For the dimensions $d_{j,i}$ of the conjugacy classes of the restrictions of M_j to the two diagonal blocks one has $d_{1,i}+\ldots+d_{p+1,i} = 2(m_i)^2$. A direct computation shows that dim $\mathrm{Ext}^1(\Phi_1,\Phi_2)=0$.

Indeed, consider the case when there are only two diagonal blocks and $M_j = \begin{pmatrix} M_j^1 & F_j \\ 0 & M_j^2 \end{pmatrix}$. Hence, there exist matrices G_j such that $F_j = M_j^1G_j - G_jM_j^2$.

One has $\dim E = 2m_1m_2$ where $E = \{(F_1,\ldots,F_{p+1})\,|\,F_j = M_j^1G_j - G_jM_j^2\}$ (to be checked by the reader). The subspace E' of E defined by

$$\sum_{j=1}^{p+1} P_j = 0\ ,\ P_j := M_1^1\ldots M_{j-1}^1F_jM_{j+1}^2\ldots M_{p+1}^2\}$$

(condition resulting from (1)) is of dimension m_1m_2. These conditions are linearly independent (the change of variables $(F_1,\ldots,F_{p+1})\mapsto(P_1,\ldots,P_{p+1})$ is bijective due to $\det M_j^i\neq 0$).

4^0. One has $\dim E' = \dim E''$ where $E'' = \{(F_1,\ldots F_{p+1})\,|\,F_j = M_j^1G - GM_j^2, G\in M_{m_1,m_2}(\mathbf{C})\}$. This is the space of blocks F of size $m_1\times m_2$ resulting from the simultaneous congugation of the $(p+1)$-tuple of matrices $M_j^0 = \begin{pmatrix} M_j^1 & 0 \\ 0 & M_j^2 \end{pmatrix}$ by a matrix $\begin{pmatrix} I & G \\ 0 & I \end{pmatrix}$.

Thus $\dim(E'/E'') = \dim\ \mathrm{Ext}^1(\Phi_1,\Phi_2)=0$.

5^0. This is true for every couple of diagonal blocks Φ_1, Φ_2. Hence, it is possible to conjugate the $(p+1)$-tuple to a block-diagonal form which contradicts the triviality of the centralizer.

6^0. Prove B). Let Φ_1 and Φ_2 have the same meaning as above. Then dim $\mathrm{Ext}^1(\Phi_1,\Phi_2)\geq 2$ and one can construct a semi-direct sum of Φ_1, Φ_2 which is not a direct one. Suppose that it is defined by matrices M_j like in 3^0. One can assume that the representations Φ_1 and Φ_2 are not equivalent (even if $m_1=m_2$) because for neither of them neither of conditions (α_{m_1}), (α_{m_2}) (which they satisfy) is an equality and there exist small deformations of the representations into nearby non-equivalent ones; when (α_n) is an equality, then such an irreducible representation is said to be *rigid*; it is unique up to conjugacy, see [Si] and [Ka].

7^0. There exist infinitesimal conjugations of the matrices M_j of the form

$$\tilde{M}_j = (I+\varepsilon X_j)^{-1}M_j(I+\varepsilon X_j)$$

such that in first approximation w.r.t. ε one has $\tilde{M}_1\ldots\tilde{M}_{p+1} = I$ and the conjugations do not result from a simultaneous infinitesimal conjugation of the matrices M_j. This follows from $d_1+\ldots+d_{p+1}\geq 2n^2+2$.

The details look like this: set $X_j = \begin{pmatrix} V_j & W_j \\ U_j & S_j \end{pmatrix}$. One can assume that $W_j=0$ because infinitesimal conjugations of M_j with the matrices $\begin{pmatrix} I & \varepsilon W_j \\ 0 & I \end{pmatrix}$ do not change the block upper-triangular form of the $(p+1)$-tuple. Hence,

$$\tilde{M}_j = M_j + \varepsilon\begin{pmatrix} [M_j^1,V_j]+F_jU_j & F_jS_j - V_jM_j^2 \\ M_j^2U_j - U_jM_j^1 & [M_j^2,S_j]-U_jF_j \end{pmatrix} = o(\varepsilon)$$

Set

$$T=\{(U_1,\ldots,U_{p+1})|U_j\in M_{m_2,m_1}(\mathbf{C}),\sum_{j=1}^{p+1}Y_j=0\}$$

where $Y_j=M_1^2\ldots M_{j-1}^2U_jM_{j+1}^1\ldots M_{p+1}^1$ (the condition $\sum_{j=1}^{p+1}Y_j=0$ is the condition $\tilde{M}_1\ldots\tilde{M}_{p+1}=I$ restricted to the left lower $m_2\times m_1$-block and considered in first approximation w.r.t. ε). One has $\dim T\geq m_1m_2+2$ (this results from $d_1+\ldots+d_{p+1}\geq 2n^2+2$).

The subspace T' of T defined by the condition $\mathrm{Tr}\sum_{j=1}^{p+1}F_jU_j=0$ is of codimension 1 in it, i.e. of dimension $\geq m_1m_2+1$. This is more than the size of the block U, i.e. more than the dimension of simultaneous infinitesimal conjugations with matrices $\left(\begin{array}{cc}I&0\\\varepsilon U&I\end{array}\right)$ (this is the subspace of T of the form $\{(U,\ldots,U)\}$). Hence, one can choose U_j from $T'/\{(U,\ldots,U)\}$ and after this choose V_j and S_j such that

$$\sum_{j=1}^{p+1}([M_j^1,V_j]+F_jU_j)=0\ ,\ \sum_{j=1}^{p+1}([M_j^2,S_j]-U_jF_j)=0$$

The matrices X_j define the infinitesimal deformations $\tilde{M}_j$.

8^0. There exists also a true deformation of the form

$$\tilde{M}_j=(I+\varepsilon X_j+\varepsilon^2Y_j(\varepsilon))^{-1}M_j(I+\varepsilon X_j+\varepsilon^2Y_j(\varepsilon))$$

with Y_j analytic in ε. The existence is justified by analogy with the basic technical tool and we leave the details for the reader. The triviality of the centralizer of the $(p+1)$-tuple of matrices M_j makes the implicit function theorem applicable.

Hence, for $\varepsilon\neq 0$ small enough the $(p+1)$-tuple of matrices $\tilde{M}_j$ is irreducible.

The lemma is proved.

3.3 The set $\mathcal{S}$

Fix the distinct complex numbers a_1, …, a_{p+2}. Consider the set $\mathcal{S}$ of $(p+2)$-tuples A' of matrices $(A_1,\ldots,A_{p+2})$ such that

1) $A_{p+2}=\mathcal{H}$, $\mathcal{H}$ was defined at the end of Subsection 2.3;
2) $A_1+\ldots+A_{p+2}=0$;
3) $(\sum_{j=1}^{p+1}\alpha_jA_j)|_{\kappa,\nu}=0$, $\kappa=m_0+1,\ldots,n$; $\nu=1,\ldots,m_0$;
4) $n=dn'$, $n'\in\mathbf{N}$, $d>1$ and $(d,m_0)=1$, see Subsection 1.2;
5) the Jordan normal forms of the matrices A_j are fixed.

Lemma 19 *Let the matrices A_j satisfy conditions 1) and 2). Then the monodromy operator at a_{p+2} of the fuchsian system (7) is scalar if and only if condition 3) holds (in which case it equals I).*

Proof:

Represent system (7) locally, at a_{p+2}, by its Laurent series

$$\dot{X}=(A_{p+2}/(t-a_{p+2})+B+o(1))X$$

where $B=-\sum_{j=1}^{p+1}\alpha_jA_j$ (to be checked directly). The change of variables (local, at a_{p+2}) $X\mapsto\mathrm{diag}((t-a_{p+2})^{-1},\ldots,(t-a_{p+2})^{-1},1,\ldots,1)X$ ($n-m_0$ units) brings system (7) to the form

$$\dot{X} = (A'_{p+2}/(t - a_{p+2}) + O(1))X$$

with $A'_{p+2} = B'$ where the restriction of the matrix B' to the left lower $((n - m_0) \times m_0$-block equals the one of B to it and all other entries of B' are 0. By Proposition 11, the monodromy operator M_{p+2} is scalar if and only if $B' = 0$.

The lemma is proved.

Definition. Call *canonical change of the eigenvalues of the matrices A_j (CCE)* a change under which each eigenvalue changes by an integer, equal eigenvalues remain equal (hence, canonical or strongly generic eigenvalues remain such), the eigenvalues of A_{p+2} (which are not canonical) do not change and the sum of all eigenvalues remains 0.

Proposition 20 *Let for given Jordan normal forms of the matrices A_j and given set $\tilde{\lambda}$ of generic eigenvalues there exist $(p+2)$-tuples from $\mathcal{S}$. Then for all generic eigenvalues sufficiently close to $\tilde{\lambda}$ there exist $(p+2)$-tuples from $\mathcal{S}$ with the same Jordan normal forms of the matrices A_j.*

The proposition is proved in the next subsection. Its proof describes a way to construct $(p+2)$-tuples from $\mathcal{S}$ for nearby eigenvalues by deforming given $(p+2)$-tuples from $\mathcal{S}$ (with given eigenvalues). This way is called the *modified basic technical tool*.

Denote by $\mathbf{C}'$ the space of eigenvalues of the matrices $A_1, \ldots, A_{p+1}$ when their Jordan normal forms are fixed.

Corollary 21 *If for given Jordan normal forms of the matrices A_j the set $\mathcal{S}$ is not empty, then for all eigenvalues from some Zariski open dense subset of $\mathbf{C}'$ there exist $(p+2)$-tuples from $\mathcal{S}$.*

The corollary follows directly from the proposition.

Denote by λ^0 a point from $\mathbf{C}'$ defining for $j \leq p+1$ canonical eigenvalues. Consider all points from $\mathbf{C}'$ obtained from λ^0 as a result of a CCE. Denote their set by $\Sigma(\lambda^0)$.

Choose λ^0 such that

a) for $j \leq p+1$ (one of) the eigenvalue(s) of A_j of greatest multiplicity is integer; denote it by λ_j^0; all other eigenvalues are non-integer (for $j \leq p+1$);

b) the only non-genericity relations modulo $\mathbf{Z}$ satisfied by the eigenvalues $\lambda_{k,j}$ are of the form

$$k(\sum_{j=1}^{p+1} \lambda_j^0) + \delta = 0 \ , \ \delta \in \mathbf{Z}$$

(the eigenvalues of A_{p+2} are integer, so we include them in δ; $k \in \mathbf{N}$ does not exceed the smallest of the multiplicities of the eigenvalues λ_j^0).

Corollary 22 *If $\mathcal{S}$ is non-empty, then the set $\Sigma(\lambda^0)$ contains a point λ^1 for which the sum $\sum_{j=1}^{p+1} \lambda_j^1$ (λ_j^1 being an integer eigenvalue of A_j) is > 1.*

Proof:

1^0. Call a point from $\Sigma(\lambda^0)$ *good* (*bad*) if $\mathcal{S}$ projects on this point (if not). If a line $l_1 \subset \mathbf{C}'$ passing through two points from $\Sigma(\lambda^0)$ contains infinitely many bad points, then it contains only bad points from $\Sigma(\lambda^0)$ (see the proposition and the corollary and remember that $\mathcal{S}$ is constructible).

2^0. Suppose that no line parallel to l_1 and passing through two points from $\Sigma(\lambda^0)$ contains only finitely many bad points. Hence, $\Sigma(\lambda^0)$ must contain only bad points. The above proposition and corollary imply that $\mathcal{S}$ does not project on any line passing through two points of $\Sigma(\lambda^0)$. Hence, it does not project on any point of any affine subspace of $\mathbf{C}'$ of dimension k passing through k points from $\Sigma(\lambda^0)$ for $k = 2, \ldots$, $\dim\mathbf{C}'$ (proved by induction on k). Hence, $\mathcal{S}$ must be empty – a contradiction.

3^0. This means that for every line l_1 passing through two points from $\Sigma(\lambda^0)$ there exists a line l_1' parallel to it, passing through two points from $\Sigma(\lambda^0)$ and containing only finitely many bad points. Choose an index j such that A_j has at least two different eigenvalues (λ_j' and λ_j'', of multiplicities m' and m''); one of the two eigenvalues is the integer eigenvalue of A_j.

4^0. Denote by t_1, t_2 two points from $\mathbf{C}'$ such that the CCE which changes t_1 to t_2 is of the form $\lambda_j' \mapsto \lambda_j' - m''$, $\lambda_j'' \mapsto \lambda_j'' + m'$ (all other eigenvalues remaining the same). Hence, there exists a line l in $\mathbf{C}'$ parallel to the one passing through t_1 and t_2, also passing through two points from $\Sigma(\lambda^0)$ and such that l contains only finitely many bad points. Hence, the line l contains a point for which the sum $\sum_{j=1}^{p+1} \lambda_j^1$ is a positive integer.

The corollary is proved.

3.4 The modified basic technical tool

The *modified basic technical tool* is used to prove the existence of deformations (depending analytically on $\varepsilon \in (\mathbf{C}, 0)$) of $(p+2)$-tuples $A' \in \mathcal{S}$ with trivial centralizers. It will be used in different contexts and we explain it here in one of them (namely, the proof of Proposition 20). The basic points in the reasoning in all other contexts will be the same, there will be differences only in the technical details.

Proof of Proposition 20:

1^0. Denote by $A_j^0 = Q_j^{-1} G_j Q_j$ the matrices from an irreducible $(p+2)$-tuple $A' \in \mathcal{S}$, G_j being Jordan matrices. We look for matrices of the form $A_{p+2} \equiv A_{p+2}^0 = H$ (H was defined in Section 2), and for $j \leq p+1$

$$A_j(\varepsilon) = (I + \varepsilon X_j(\varepsilon))^{-1} Q_j^{-1} (G_j + \varepsilon L_j) Q_j (I + \varepsilon X_j(\varepsilon)) \ , \ \varepsilon \in (\mathbf{C}, 0)$$

where L_j are diagonal matrices which are polynomials of the semi-simple parts of the corresponding matrices G_j. Hence, $[G_j, L_j] = 0$ and the matrices G_j and $G_j + \varepsilon L_j$ define one and the same Jordan normal form for ε small enough, see 4) – 5) from Theorem 14. We want conditions 1) – 5) from the previous subsection to hold.

Obviously, one has $A_j = A_j^0 + \varepsilon([A_j^0, X_j(0)] + Q_j^{-1} L_j Q_j) + o(\varepsilon)$ for $j \leq p+1$.

2^0. Conditions 2) and 3) yield in first approximation w.r.t. ε the following system of equations linear in the unknown variables the entries of the matrices $X_j(0)$:

$$\sum_{j=1}^{p+1} [A_j^0, X_j(0)] = -\sum_{j=1}^{p+1} Q_j^{-1} L_j Q_j \quad , \quad (\sum_{j=1}^{p+1} \alpha_j [A_j^0, X_j(0)])|_{\kappa,\nu} = -(\sum_{j=1}^{p+1} \alpha_j Q_j^{-1} L_j Q_j)|_{\kappa,\nu} \qquad (10)$$

with $\kappa = m_0 + 1, \ldots, n$; $\nu = 1, \ldots, m_0$; $\alpha_j = 1/(a_j - a_{p+2})$. The left hand-sides of these equations are linear forms in the entries of the matrices $X_j(0)$.

Lemma 23 *These linear forms are linearly independent.*

The proof of this lemma occupies the rest of the proof of the proposition. It implies the existence of X_j analytic in $\varepsilon \in (\mathbf{C}, 0)$. Indeed, for $\varepsilon = 0$ equations (10) are solvable and the mapping

$$(X_1(0), \ldots, X_{p+1}(0)) \mapsto (\sum_{j=1}^{p+1}[A_j^0, X_j(0)], (\sum_{j=1}^{p+1} \alpha_j[A_j^0, X_j(0)])|_{\kappa,\nu})$$

is surjective onto $sl(n, \mathbf{C}) \times \mathbf{C}^{(n-m_0)m_0}$. The existence of X_j analytic in ε small enough follows from the implicit function theorem.

Proof of the lemma:

1^0. If the lemma were not true, then there should exist a couple of matrices $(0, 0) \neq (V, W) \in sl(n, \mathbf{C})^2$ such that $W_{i,j} = 0$ if $i > m_0$ or if $j \leq m_0$ and

$$\mathrm{tr}(V(\sum_{j=1}^{p+1}[A_j^0, X_j(0)]) + W(\sum_{j=1}^{p+1} \alpha_j[A_j^0, X_j(0)])) = 0 \quad \text{for all } X_j(0), \text{ i.e.}$$

$\mathrm{tr}(([V, A_j^0]+\alpha_j[W, A_j^0])X_j(0)) = 0$ identically in the entries of $X_j(0)$, i.e. $[V, A_j^0]+\alpha_j[W, A_j^0] = 0$.

Summing up the equalities for $j = 1, \ldots p+1$ and making use of $\sum_{j=1}^{p+1} A_j^0 = -A_{p+2}^0$, one gets

$$[V, A_{p+2}^0] + [W, -\sum_{j=1}^{p+1} \alpha_j A_j^0] = 0 \tag{11}$$

The left lower $m_0 \times (n - m_0)$-block of the second summand in (11) is 0. Hence, so is the left lower block of the first summand, and, hence, the one of V itself (remember that $A_{p+2}^0 = \mathcal{H}$).

2^0. Choose $\gamma \in \mathbf{C}$ such that the matrix $V' = V + \gamma I$ be non-degenerate. Hence, for all $t \in \mathbf{C} \backslash \{a_{p+2}\}$ the matrix $V' + W/(t - a_{p+2})$ is non-degenerate, block upper-triangular and $\det(V' + W/(t - a_{p+2})) \equiv \det V'$. Hence, its inverse is also block upper-triangular, with constant diagonal blocks and with constant non-zero determinant.

3^0. Consider the fuchsian system $\dot{X} = A(t)X$ with $A(t) = (\sum_{j=1}^{p+2} A_j^0/(t - a_j))$. Perform in it the change of variables $X \mapsto (V' + W/(t - a_{p+2}))X$:

$$A(t) \to (V' + W/(t - a_{p+2}))^{-1}(W/(t - a_{p+2})^2) + (V' + W/(t - a_{p+2}))^{-1}A(t)(V' + W/(t - a_{p+2}))$$

(gauge transformation). The matrix $U(t) \stackrel{\mathrm{def}}{=} V' + W/(t - a_{p+2})$ being holomorphic and holomorphically invertible for $t \neq a_{p+2}$, the system remains fuchsian at $a_1, \ldots, a_{p+1}$ and has no singularities other than a_j.

The equalities $[\gamma I+V, A_j^0]+\alpha_j[W, A_j^0] = 0$ are equivalent to $[U(a_j), A_j^0] = 0$ and imply that its residua at a_j, $j \leq p+1$, don't change. (The residua of the new system equal $(U(a_j))^{-1}A_j^0 U(a_j)$.)

Check that the system remains fuchsian at a_{p+2} as well. To this end observe that the matrix $U^{-1} = (V'(I + (V')^{-1}W/(t - a_{p+2}))^{-1}$ equals $(V')^{-1} - (V')^{-1}W(V')^{-1}/(t - a_{p+2})$ because $(I + (V')^{-1}W/(t - a_{p+2}))^{-1} = I - (V')^{-1}W/(t - a_{p+2})$ due to $((V')^{-1}W)^2 = 0$ (recall the block upper-triangular form of V' and W). This implies that there are no polar terms of order higher than 1 at a_{p+2} in the new system.

Indeed, a priori the matrix $-U^{-1}\dot{U} + U^{-1}A(t)U$ (with U and U^{-1} as above) can have at a_{p+2} a pole of order ≤ 3. Its coefficient before $1/(t - a_{p+2})^3$ equals

$$-((V')^{-1}W)^2 - (V')^{-1}W(V')^{-1}A_{p+2}W$$

where each of the two summands is 0 (this follows from the form of the matrices V, W and A_{p+2}). The one before $1/(t-a_{p+2})^2$ equals

$$(V')^{-1}W-(V')^{-1}W(V')^{-1}A_{p+2}V'+(V')^{-1}A_{p+2}W=(V')^{-1}PV' ,$$

$$P=W(V')^{-1}-W(V')^{-1}A_{p+2}+A_{p+2}W(V')^{-1}$$

where $P=0$ for the same reason. Hence, the residuum at a_{p+2} of the system also doesn't change (because the sum of all residua is 0).

4^0. Hence, the solution UX to the new system (which is the old one) equals XC, $C\in GL(n,\mathbf{C})$ (some solution to the old system). Make the analytic continuations of both solutions along one and the same closed contour. Hence, $UXM=XMC$ where M is the monodromy operator corresponding to the contour. But $UXM=XCM$ which implies $[C,M]=0$. This is true for any closed contour.

The monodromy group of the system being irreducible, the matrix C must be scalar. Hence one has $UX\equiv CX$, i.e. $U\equiv C$ which implies $V'=C$, $W=0$. Recall that $V'=V+\gamma I$, $V\in sl(n,\mathbf{C})$. Hence, $V=0$.

The lemma is proved, the proposition as well.

4 Proof of the necessity

4.1 The proof in the case when ξ is primitive

Given a $(p+1)$-tuple with a trivial centralizer, one can deform it into one also with a trivial centralizer, with relatively generic eigenvalues and with the same Jordan normal forms, therefore we presume the eigenvalues relatively generic.

We make use of Lemma 18 and Corollary 15 and consider only the case when all matrices A_j and M_j are diagonalizable. Denote by Λ^n the PMVs of their eigenvalues; they are defined by the Jordan normal forms J_j^n. Denote by Λ^{n_ν} the PMVs defined by the Jordan normal forms $J_j^{n_\nu}$ (their definition is given before Theorem 1).

Consider the fuchsian system

$$\dot{X}=(\sum_{j=1}^{p+2}A_j/(t-a_j))X \qquad (12)$$

where the $(p+2)$-tuple of matrices A_j belongs to $\mathcal{S}$ and the eigenvalues of A_j satisfy the conclusion of Corollary 22. Hence, the $(p+2)$-tuple of matrices A_j is irreducible and the eigenvalues of its monodromy operators satisfy the only non-genericity relation

$$\sigma_1\ldots\sigma_{p+2}=1 \qquad (\gamma_0)$$

where $\sigma_j=1$ (notice that 1 is the only eigenvalue of M_{p+2}).

Lemma 24 *The monodromy group of system (12) with eigenvalues defined as above has a trivial centralizer.*

The eigenvalues of system (12) satisfy the conditions of Lemma 12 and Lemma 24 follows from Corollary 13.

Lemma 25 *The monodromy group of system (12) with eigenvalues defined as in Lemma 24 can be conjugated to the form* $\left(\begin{array}{cc}\Phi & *\\ 0 & I\end{array}\right)$ *where* Φ *is* $n_1 \times n_1$.

Remark: Notice that the subrepresentation Φ can be reducible.

Lemma 26 *The centralizer* $Z(\Phi)$ *of the subrepresentation* Φ *is trivial.*

The last two lemmas are proved in Subsection 4.3. We prove them also in the case when ξ is not primitive because we need this for the next subsection.

The subrepresentation Φ being of dimension $n_1 < n$, one can use induction on n to prove the necessity. The induction base is the case when condition (ω_n) holds – in this case there is nothing to prove.

The PMV of the matrices M_j' defining Φ equals Λ^{n_1}. It follows from Lemma 26 that for generic eigenvalues close to the ones of the matrices M_j' defining Φ there exist irreducible $(p+1)$-tuples of diagonalizable matrices $\tilde{M}_j' \in GL(n_1, \mathbf{C})$ with PMV Λ^{n_1} and satisfying (1) (this can be proved by using the basic technical tool in the multiplicative version).

Hence, if Λ^n is good, then Λ^{n_1} is good. Condition (ω_n) doesn't hold by assumption and conditions (α_n) and (β_n) hold, see the Introduction. Finally, the PMV Λ^{n_s} is the same for Λ^n and for Λ^{n_1} (this follows from the definition of the PMVs Λ^{n_ν}). If Λ^{n_1} is good, then Λ^{n_s} satisfies condition (ω_{n_s}) (one can't have $n_s = 1$, see the remark after Theorem 1). Hence, if the PMV Λ^n is good, then it satisfies conditions *i)* and *ii)* of Theorem 1, i.e. they are necessary.

The necessity is proved.

4.2 The proof in the case when ξ is not primitive and in the additive version

1^0. If ξ is not primitive, then the proof needs only small modifications. The eigenvalues of the matrices A_j of system (12) are only relatively generic and satisfy only the non-genericity relation (γ^*) defined at the end of Subsection 1.2 (and its multiples).

2^0. Hence, the $(p+1)$-tuple of matrices-residua A_j might be reducible. Suppose that it is in block upper-triangular form. Consider instead of the system its restriction to one of the diagonal blocks P; this restriction is presumed to be irreducible.

For this restriction Lemma 24 holds again. This follows from Corollary 13. The rest of the proof is the same because for all j J_j^n is a multiple of $J(A_j|_P)$. Recall that Lemmas 25 and 26 are proved also in the case when ξ is not primitive.

3^0. In the additive version we proved the necessity in the case of generic eigenvalues in [Ko1]. If the eigenvalues are non-generic and if $q = 1$, then given such an irreducible $(p+1)$-tuple of matrices A_j one can deform it by means of the basic technical tool into a nearby one with generic eigenvalues and the same Jordan normal forms of the respective matrices. Thus the necessity is proved for $q = 1$ in the additive version.

4^0. Consider the additive version with $q > 1$. Given such an irreducible $(p+1)$-tuple of matrices A_j, one can multiply it by $c \in \mathbf{C}^*$ to make the eigenvalues canonical. Next, by means of the basic technical tool one can deform it into a nearby irreducible one with the same Jordan normal forms of the matrices A_j and with canonical relatively strongly generic eigenvalues. By Corollary 10, the monodromy group must be irreducible. By Proposition 11, for each j one has $J(A_j) = J(M_j)$. Conditions *i)* and *ii)* are necessary in the multiplicative version, therefore they will be fulfilled in the additive one as well.

4.3 Proofs of Lemmas 25 and 26

Proof of Lemma 25:

We assume that condition (ω_n) does not hold (otherwise there is nothing to prove).

1^0. Consider first (in $1^0 - 5^0$) the case when ξ is a primitive root of unity. The monodromy group can be conjugated to a block upper-triangular form. The diagonal blocks define either irreducible or one-dimensional representations. The eigenvalues of each diagonal block 1×1 satisfy the non-genericity relation (γ_0); it is the only one satisfied by them due to the definition of the eigenvalues. This means that there is a single diagonal block of size > 1.

2^0. The block in the right lower corner must be of size 1. Indeed, by Lemma 9 the left upper block can't be of size 1 (because the corresponding sum of eigenvalues $\lambda_{k,j}$ is a positive integer). Hence, it must be the only block of size > 1 and the matrices M_j look like this:

$$M_j = \left(\begin{array}{cc} M_j' & L_j \\ 0 & I \end{array} \right) \qquad (M)$$

where the size of M_j' is $n' \times n'$ and the $(p+1)$-tuple of matrices M_j' is irreducible.

3^0. The block M' must be of size $\leq n_1$. Indeed, if its size n' is $> n_1$, then we show that the columns of the $(p+1)$-tuples of matrices L_j aren't linearly independent modulo the space $\mathcal{W}$ defined below which will imply the non-triviality of the centralizer of the monodromy group.

Denote by $\mathcal{W} \subset \mathbf{C}^{n'}$ the space of $(p+1)$-tuples of vector-columns of the form $(M_j - I)X$, $X \in \mathbf{C}^{n'}$. These are the vector-columns (right upper blocks) obtained by conjugating the $(p+1)$-tuple of matrices $\left(\begin{array}{cc} M_j' & 0 \\ 0 & 1 \end{array} \right)$ by $\left(\begin{array}{cc} I & X \\ 0 & 1 \end{array} \right)$.

One has dim$\mathcal{W} = n'$. Indeed, if dim$\mathcal{W} < n'$, then the images of the linear operators $X \mapsto (M_j - I)X$ belong to a proper subspace of $\mathbf{C}^{n'}$. This subspace can be assumed to belong to the space spanned by the first $n'-1$ vectors of the canonical basis of $\mathbf{C}^{n'}$ (which can be achieved by conjugating the monodromy operators M_j by a block-diagonal matrix with diagonal blocks of sizes n' and $n - n'$, the latter equal to I).

But then the matrices M_j' will be of the form $\left(\begin{array}{cc} M_j'' & * \\ 0 & 1 \end{array} \right)$ which contradicts the irreducibility of the matrix group generated by M_1', $\ldots$, M_{p+1}'.

4^0. The $(p+1)$-tuples of columns of the blocks L_j belong to a subspace of $\mathbf{C}^{(p+1)n'}$ of dimension $\Delta := r_1 + \ldots + r_{p+1} - n'$ (each column of L_j belongs to a space of dimension r_j and there are n' linear conditions satisfied by these columns; these conditions result from (1) and look like this: $\sum_{j=1}^{p+1} M_1' \ldots M_{j-1}' L_j = 0$; they are linearly independent because the change of variables $L_j' = M_1' \ldots M_{j-1}' L_j$ transforms them to $\sum_{j=1}^{p+1} L_j' = 0$ and the independence in this form is evident).

5^0. The columns of the block L must be linearly independent modulo the space $\mathcal{W}$, otherwise the monodromy group will be a direct sum of the form $M_j = \left(\begin{array}{cc} M_j''' & 0 \\ 0 & 1 \end{array} \right)$, $M_j''' \in GL(n-1, \mathbf{C})$. Hence, $\Delta \geq (n - n') + n'$ (because the block L has $n - n'$ columns and dim$\mathcal{W} = n'$). Recall that $r_1 + \ldots + r_{p+1} = n + n_1$. Hence, $n + n_1 - n' \geq n - n' + n'$, i.e. $n' \leq n_1$.

6^0. Let now ξ be a non-primitive root. The diagonal blocks of the monodromy group can be of two types. The first are of size 1, the eigenvalues satisfying the non-genericity relation (γ_0).

Describe the second type of diagonal blocks. Their sizes are > 1 and can be different. Define the *unitary set* of eigenvalues: for each j divide by (m_0, q) the multiplicities of all eigenvalues $\sigma_{k,j}$ of the ones that are equal among themselves and are $\neq 1$. A block F of the second type

contains h times the unitary set, $1 \leq h \leq (m_0, q)$, and a certain number of eigenvalues equal to 1. (To different matrices M_j there correspond, in general, different numbers of eigenvalues from the unitary set; therefore one must, in general, add some number of eigenvalues 1 for some values of j to make the number of eigenvalues of the restrictions of the matrices M_j to F equal; one then could eventually add one and the same number of eigenvalues equal to 1 to all matrices $M_j|_F$.)

The eigenvalues of the blocks of the second type satisfy the non-genericity relation (γ^*) (defined in Subsection 1.2) and eventually (γ_0) as well.

7^0. Denote by $\kappa(F)$ the ratio "number of eigenvalues $\sigma_{k,j}$ equal to 1"/"number of eigenvalues $\sigma_{k,j}$ not equal to 1" (eigenvalues of the restriction of the monodromy group to F), and by κ_0 the same ratio computed for the entire matrices M_j (in both ratios one takes into account the eigenvalues of all matrices M_j). Then one must have $\kappa(F) < \kappa_0$.

Indeed, one can't have $\kappa(F) \geq \kappa_0$ because condition (ω_n) does not hold, hence, the restriction of the monodromy group to F wouldn't satisfy this condition either. In the presence of the non-genericity relation (γ_0) this implies a contradiction with the following

Lemma 27 *(see [Ko1]). The following condition is necessary for the existence of irreducible $(p+1)$-tuples of matrices M_j from the conjugacy classes C_j and satisfying (1):*

$$\min_{b_1,\ldots,b_{p+1}\in\mathbf{C}^*,b_1\ldots b_{p+1}=1}(\mathrm{rk}(b_1M_1-I)+\ldots+\mathrm{rk}(b_{p+1}M_{p+1}-I))\geq 2n$$

But then the sum of the eigenvalues $\lambda_{k,j}$ corresponding to the eigenvalues $\sigma_{k,j}$ from F will be negative. If the block F is to be in the right lower corner, then this sum must be positive (Lemma 9; it can't be 0 because the only non-genericity relation satisfied by the eigenvalues $\lambda_{k,j}$ is (γ^*) and its multiples, see Subsection 1.2; if the sum is 0, then by Corollary 10 the $(p+1)$-tuple of matrices-residua would be block upper-triangular up to conjugacy – a contradiction). Hence, the right lower block is of size 1.

8^0. Denote by Π the left upper $(n-1)\times(n-1)$-block. Conjugate it to make all non-zero rows of the restriction of the $(p+1)$-tuple $\tilde{M}$ of matrices $M_j - I$ to Π linearly independent. After the conjugation some of the rows of the restriction of $\tilde{M}$ to Π might be 0. In this case conjugate the matrices M_j by one and the same permutation matrix which places the zero rows of $M_j - I$ in the last (say, $n-n'$) positions (recall that the last row of $M_j - I$ is 0, so $n-n' \geq 1$). Notice that if the restriction to Π of a row of $M_j - I$ is zero, then the n-th position of the row is 0 as well, otherwise M_j is not diagonalizable.

9^0. After this conjugation the monodromy matrices have the form (M) from 2^0, the matrix group generated by the matrices M_j' is not necessarily irreducible, but can be conjugated to a block upper-triangular form, its restrictions to the diagonal blocks (all of sizes > 1) being irreducible. Hence, the mapping

$$(X_1,\ldots,X_{p+1})\mapsto\sum_{j=1}^{p+1}(M_j'-I)X_j\ ,\ X_j\in\mathbf{C}^{n'}\tag{13}$$

is surjective onto $\mathbf{C}^{n'}$. Indeed, the matrix algebra $\mathcal{M}$ generated by the matrices $M_j' - I$ is block upper-triangular, its restrictions to each diagonal block (say, of size u) is irreducible and, hence, is $gl(u,\mathbf{C})$ (the Burnside theorem). Thus the algebra contains a non-degenerate matrix L.

One has $L = \sum_{j=1}^{p+1}(M_j'-I)H_j$, $H_j \in \mathcal{M}$. For every vector-column $X \in \mathbf{C}^{n'}$ there exists a unique $Y \in \mathbf{C}^{n'}$ such that $X = LY$. Hence, one can set $X_j = H_jY$ which proves the surjectivity of the mapping.

10^0. If one defines the space $\mathcal{W}$ as above, then one finds that $\dim\mathcal{W} = n'$. This is proved like in the case when ξ is a primitive root, see 3^0, but the form $\left(\begin{array}{cc} M_j'' & * \\ 0 & 1 \end{array}\right)$ of the matrices M_j' is forbidden not because the group generated by them must be irreducible (which, in general, is not true) but just because by definition there are no diagonal blocks of size 1. For the rest the proof goes on like in the case when ξ is primitive.

This proves the lemma.

Proof of Lemma 26:

1^0. Consider first the case when ξ is a primitive root (in $1^0 - 4^0$). If the lemma is not true, then $Z(\Phi)$ either A) contains a diagonalizable matrix D with exactly two distinct eigenvalues or B) it contains a nilpotent matrix N with $N^2 = 0$, see 1^0 of the proof of Corollary 13.

2^0. In case A) one conjugates the monodromy group to the form $\left(\begin{array}{cc} G_j & L_j \\ 0 & I \end{array}\right)$ with $G_j = \left(\begin{array}{cc} M_j^1 & 0 \\ 0 & M_j^2 \end{array}\right)$. The sizes of M_j^1, M_j^2 equal the multiplicities of the two eigenvalues of D and one has $D = \left(\begin{array}{cc} \alpha I & 0 \\ 0 & \beta I \end{array}\right)$, $\alpha \neq \beta$, D and G_j are $n_1 \times n_1$.

At least one of the blocks M_j^i must equal I (for all j) because there is a single diagonal block of size > 1. But this would mean that the monodromy group is a direct sum. Indeed, if $M_j^2 = I$ for all j, then in the rows of the block M_j^2 and in the last columns (the ones of the block I) the entries of M_j must be 0, otherwise M_j is not diagonalizable. Being a direct sum contradicts Lemma 24.

3^0. In case B) one can conjugate N to the form $N = \left(\begin{array}{ccc} 0 & 0 & I \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right)$ (or $N = \left(\begin{array}{cc} 0 & I \\ 0 & 0 \end{array}\right)$) with I being $v \times v$, $v \leq n_1/2$; the second case corresponds to $v = n_1/2$.

Hence, $M_j = \left(\begin{array}{cccc} M_j^1 & R_j & T_j & Q_j \\ 0 & M_j^2 & S_j & H_j \\ 0 & 0 & M_j^1 & P_j \\ 0 & 0 & 0 & I \end{array}\right)$ where M_j^1 is $v \times v$ and if $v = n_1/2$, then the blocks R_j, M_j^2, S_j and H_j are absent.

4^0. Like in case A) one shows that $M_j^1 = I$ (this implies that in fact the possibility $v = n_1/2$ does not exist because there would be no diagonal block of size > 1 at all). This means that the matrix with a single non-zero entry in position $(1, n)$ belongs to the centralizer of the monodromy group which contradicts Lemma 24.

5^0. Let ξ not be a primitive root. Suppose that we are in case A). We showed in the proof of the previous lemma (see 4^0 of its proof) that the $(p+1)$-tuple of columns of the blocks L_j (see 2^0 of the present proof) belong to a subspace of $\mathbf{C}^{(p+1)n_1}$ (denote it by $\mathcal{V}$) of dimension $\Delta = n + n_1 - n_1 = n$. When this subspace is factorized by the space $\mathcal{W}$ (see 3^0 of the proof of the previous lemma), then the dimension becomes $n - n_1$.

On the other hand, there are $n - n_1$ columns of the blocks L_j. In case A) the space $\mathcal{V}/\mathcal{W}$ splits into a direct sum of two such spaces defined for each of the blocks M_j^1 and M_j^2. The sum of their dimensions equals $n - n_1$, the number of columns of L_j; this implies that one can conjugate the matrices M_j by $G \in GL(n, \mathbf{C})$ to the form $M_j = \left(\begin{array}{cccc} M_j^1 & 0 & L_j' & 0 \\ 0 & M_j^2 & 0 & L_j'' \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{array}\right)$ which means that

the monodromy group is a direct sum (one makes a self-evident permutation of the rows and columns which results from conjugation to achieve a block-diagonal form of the matrices). The matrix G is block-diagonal, with diagonal blocks of sizes n_1 and $n-n_1$, the former equal to I.

This is a contradiction with Lemma 24.

6^0. In case B) a conjugation with a matrix G' (defined like G in 5^0) brings the block $\left(\begin{array}{c} Q_j \\ H_j \\ P_j \end{array}\right)$ to the form $\left(\begin{array}{ccc} U_j^1 & 0 & 0 \\ 0 & U_j^2 & 0 \\ 0 & 0 & U_j^1 \end{array}\right)$ with heights of the blocks U_j^1, U_j^2 the same as the ones of M_j^1, M_j^2.

This means that the matrix $\left(\begin{array}{cccccc} 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$ belongs to the centralizer of the monodromy group. This is again a contradiction with Lemma 24.

The lemma is proved.

5 Proof of the sufficiency

5.1 Plan of the proof

1^0. Two cases are possible:

Case A) $d_1+\ldots+d_{p+1}\geq 2n^2+2$

Case B) $d_1+\ldots+d_{p+1}=2n^2$

The condition $d>1$ excludes the possibility $d_1+\ldots+d_{p+1}=2n^2-2$, see the remarks after Theorem 1. In case B) ξ is presumed primitive.

In case A) we construct $(p+1)$-tuples of nilpotent matrices A_j with trivial centralizers where for each j $J(A_j)$ corresponds to the necessary Jordan normal form $J(M_j)$ of the monodromy operator M_j, see Lemma 28, part I). Such a construction was already carried out in [Ko1].

2^0. After this we deform the $(p+1)$-tuple into a nearby $(p+2)$-tuple of matrices $\tilde{A}_j$ with $J(\tilde{A}_j)=J(M_j)$ for $j\leq p+1$, with $A_{p+2}=\varepsilon\mathcal{H}$ and with

$$\tilde{A}_j=(I+\varepsilon X_j(\varepsilon))^{-1}Q_j^{-1}(G_j+\varepsilon L_j)Q_j(I+\varepsilon X_j(\varepsilon))$$

for $j=1,\ldots,p+1$ so that the matrix $G_j+\varepsilon L_j$ be conjugate to εL_j (see 6) of Theorem 14). We do this like in the modified basic technical tool. The following condition must hold:

$$(\sum_{j=1}^{p+1}\tilde{A}_j/(a_j-a_{p+2}))_{\mu,\nu}=0\ ,\ \mu=m_0+1,\ldots,n\ ,\ \nu=1,\ldots,m_0. \qquad (14)$$

3^0. Then one multiplies the $(p+2)$-tuple by $1/\varepsilon$; hence, the difference between the two eigenvalues of $\tilde{A}_{p+2}$ becomes equal to 1. Condition (14) implies that the singularity at a_{p+2} of the fuchsian system with residua $\tilde{A}_j$ will be apparent (i.e. with local monodromy equal to I).

4^0. In case B) one constructs $(p+2)$-tuples of nilpotent matrices A_j with trivial centralizers, see Lemma 28, part II). The Jordan normal form of the matrix A_{p+2} has m_0 blocks of size 2

and $n-2m_0$ ones of size 1. One sets $\tilde{A}_{p+2}=A_{p+2}+\varepsilon\mathcal{H}$, the other matrices $\tilde{A}_j$ are defined like in case A). The matrix A_{p+2} is such that for $\varepsilon\neq 0$ the matrix $\tilde{A}_{p+2}$ is conjugate to $\mathcal{H}$. One has $\tilde{A}_{p+2}|_{\varepsilon=0}\neq 0$ to make possible the construction of a $(p+2)$-tuple of matrices A_j with a trivial centralizer.

Like in case A) one multiplies the residua $\tilde{A}_j$ by $1/\varepsilon$. Condition (14) holds in case B) as well and the fuchsian system obtained after this multiplication has an apparent singularity at a_{p+2}.

5^0. The deformation of A_j into $\tilde{A}_j$ is possible to be done when certain transversality conditions make the theorem of the implicit function applicable.

5.2 The basic lemma

Denote by $\tilde{J}_j^n$ Jordan normal forms ($j=1,\ldots,p+1$) satisfying conditions *i)* and *ii)* of Theorem 1. Denote by J_j^n both their corresponding Jordan normal forms with a single eigenvalue and the nilpotent conjugacy classes defining these Jordan normal forms.

Lemma 28 *Let $d>1$. Let the $p+1$ Jordan normal forms J_j^n satisfy conditions* i) *and* ii) *of Theorem 1.*

I) Then in case A) there exists a $(p+1)$-tuple of nilpotent matrices A_j satisfying (2) such that

1) for $j=1,\ldots,p+1$ the matrix A_j belongs either to J_j^n or to its closure;

2) the centralizer $\mathcal{Z}$ of the $(p+1)$-tuple is trivial;

3) the $(p+1)$-tuple is in block upper-triangular form, the diagonal blocks being of sizes n_s, $n_{s-1}-n_s$, …, $n-n_1$; the restriction of the $(p+1)$-tuple only to the first of them is non-zero;

4) the restriction to the diagonal block of size n_s of the matrix $B:=\sum_{j=1}^{p+1}\alpha_jA_j$ is in Jordan normal form and its first d eigenvalues are non-zero and simple;

5) the restrictions to this block of the matrices A_j define an irreducible representation.

II) In case B) there exists a $(p+2)$-tuple of nilpotent matrices A_j satisfying (2) such that 1), 2) and 3) hold and

6) the Jordan normal form of the matrix A_{p+2} consists of m_0 blocks of size 2 and of $n-2m_0$ blocks of size 1; its non-zero entries are all in the diagonal block of size n_s;

7) the restrictions to the diagonal block of size n_s of the matrices A_j are themselves block upper-triangular (and define a representation with a trivial centralizer), with diagonal blocks of equal size (which is 2, 3, 4 or 6) defining non-equivalent irreducible representations;

8) the restriction to the diagonal block of size n_s of the matrix B is upper-triangular, with distinct non-zero eigenvalues.

Remark: The Jordan normal forms $J_j^{n_s}$ from case B) correspond to a triple or quadruple of matrices from the four so-called *special cases*, see Subsection 6.1.

Corollary 29 *Let the $(p+1)$-tuple of conjugacy classes $C_j\in GL(n,\mathbf{C})$ define Jordan normal forms J_j^n which satisfy conditions* i) *and* ii) *of Theorem 1. Let $d>1$ and let in case B) ξ be primitive. Then there exist $(p+1)$-tuples of matrices $M_j\in C_j$ satisfying (1) and with trivial centralizers.*

Proof:

1^0. Construct $(p+1)$- or $(p+2)$-tuples of nilpotent matrices A_j^0 like in the lemma where for $j=1,\ldots,p+1$ A_j^0 is nilpotent and $J(A_j^0)=J_j^n$. Set $A_j^0=Q_j^{-1}G_jQ_j$ where G_j are nilpotent Jordan matrices. Then construct matrices

$$\tilde{A}_j = (I + \varepsilon X_j(\varepsilon))^{-1} Q_j^{-1}(G_j + \varepsilon L_j) Q_j (I + \varepsilon X_j(\varepsilon))$$

such that for $j = 1, \ldots, p+1$ the matrix $G_j + \varepsilon L_j$ be conjugate to εL_j (see 6) of Theorem 14).

2^0. Choose the eigenvalues of the matrices L_j to be canonical for $j \leq p+1$ and to have $\exp(2\pi i L_j) \in C_j$ (see Proposition 11). Set $L_{p+2} = \mathcal{H}$.

Lemma 30 *There exist matrices X_j analytic in $\varepsilon \in (\mathbf{C}, 0)$ such that*

a) $X_{p+2} \equiv 0$;

b) $\tilde{A}_1 + \ldots + \tilde{A}_{p+2} = 0$;

c) $(\sum_{j=1}^{p+1} \tilde{A}_j/(a_j - a_{p+2}))_{\mu,\nu} = 0$ for $\mu = m_0+1, n$, $\nu = 1, \ldots, m_0$.

The lemma is proved in the next subsection.

3^0. Fix $\varepsilon \neq 0$. Multiply the matrices $\tilde{A}_j$ by $1/\varepsilon$. Conditions a) and c) imply that the singularity at a_{p+2} of the fuchsian system with poles a_j and residua $\tilde{A}_j$ is apparent. Hence, the monodromy operators have the necessary eigenvalues and Jordan normal forms.

Let ξ be primitive. Choose the eigenvalues of the matrices A_j generic and satisfying the conditions of Corollary 13. Hence, the centralizer of the monodromy group is trivial. If in case A) ξ is not primitive, then the existence of representations with trivial centralizers follows from Lemma 6.

The corollary is proved.

5.3 Proof of Lemma 30

The reader is supposed to have read and understood the proof of Lemma 32 to which we refer.

1^0. Like in the proof of Proposition 20 one finds out that the existence of the necessary matrices X_j follows from the solvability of the system of linear equations (10). So one has to prove Lemma 23. Notice that this time one has $A_{p+2} = 0$, unlike in the conditions of Lemma 23, so the lemma has to be reproved. This is what we do.

2^0. In case A) equation (11) is reduced to $[W, \sum_{j=1}^{p+1} \alpha_j A_j^0] = 0$ which implies $W = 0$ (recall condition 4) from Lemma 28 and the fact that only the entries in positions (κ, ν) of W can be non-zero with $\kappa = 1, \ldots, m_0 < d$, $\nu = m_0+1, \ldots, n$).

Hence, $[V, A_j^0] = 0$ for $j = 1, \ldots, p+1$ which implies $V = 0$ (recall condition 2) from Lemma 28).

3^0. Consider case B). Denote by $H_{\mu,\nu}$ the block in position (μ, ν) when the matrices from $gl(n, \mathbf{C})$ are block-decomposed, with sizes of the diagonal blocks equal to n_s, $n_{s-1} - n_s$, $n_{s-2} - n_{s-1}$, $\ldots$, $n - n_1$. Recall that $m_0 < d < n_s$.

Equation (11) implies that $V_{H_{i,1}} = 0$ for $i > 1$. This follows from conditions 3) and 8) of Lemma 28 and from $W_{H_{i,1}} = 0$ for $i > 1$. The same equation implies that the restriction of W only to $H_{1,1}$ can be non-zero. Indeed, one has $[V, A_{p+2}^0]_{H_{1,i}} = 0$ for $i > 1$, hence, $[W, \sum_{j=1}^{p+1} \alpha_j A_j^0]_{H_{1,i}} = 0$. Conditions 3) and 8) of Lemma 28 imply that $W_{H_{1,i}} = 0$. On the other hand, for $j > 1$ one has $W_{H_{j,i}} = 0$.

4^0. We prove (in 5^0 – 8^0) that $V_{H_{1,1}} = W_{H_{1,1}} = 0$. By 3^0, this would imply $W = 0$ and the equations $[V, A_j^0] = 0$, $j = 1, \ldots, p+1$ would imply $V = 0$ (recall condition 2) of Lemma 28). Hence, $V = W = 0$ which proves Lemma 23. As we know that $V_{H_{i,1}} = W_{H_{i,1}} = 0$ for $i > 1$, we can replace in the equations

$$[V + \alpha_j W, A_j^0] = 0 \tag{15}$$

the matrices V, W and A_j^0 by their restrictions to $H_{1,1}$. This is what we do.

5^0. Block-decompose the matrices from $gl(n_s,\mathbf{C})$, all diagonal blocks being of size 2,3,4 or 6 (recall that the Jordan normal forms $J_j^{n_s}$ correspond to one of the *special cases* from Subsection 6.1). Denote the blocks by $U_{\mu,\nu}$. The form of the matrices A_{p+2} and W implies that the restriction of equation (15) to $U_{i,k}$ with $i\geq k$ looks like this:

$$V|_{U_{i,k}}A_j|_{U_{k,k}}-A_j|_{U_{i,i}}V|_{U_{i,k}}=0$$

Hence, if $i>k$, then $V|_{U_{i,k}}=0$, if $i=k$, then $V|_{U_{i,k}}=\gamma_i I$. This follows from the non-equivalence of the irreducible representations defined by the matrices $A_j|_{U_{k,k}}$ and $A_j|_{U_{i,i}}$ and from Schur's lemma.

6^0. The reader should have read the proof of Lemma 32 to which we refer. In this proof the blocks F', F'', F and ${}^tF$ were defined. Like in the proof of Lemma 32 one shows that $V_{U_{\mu,\nu}}=0$ if $\mu\neq\nu$, $U_{\mu,\nu}\not\subset F$, and $V_{U_{\mu,\nu}}=\gamma_\mu I$ if $\mu=\nu$. Let $U'=U_{\mu,\nu}\subset F$. Then one has

$$A_j|_{U_{\mu,\mu}}(V+\alpha_jW)|_{U'}-(V+\alpha_jW)|_{U'}A_j|_{U_{\nu,\nu}}+(\gamma_\mu-\gamma_\nu)A_j|_{U'}=0 \qquad (16)$$

Consider the fuchsian system with poles at $a_1,\ldots,a_{p+1}$ and residua A_j^0. Perform in it the change of variables

$$X\mapsto\tilde{W}X\ ,\ \tilde{W}=\delta I+V+W/(t-a_{p+2}) \qquad (17)$$

where $\delta\in\mathbf{C}$ is chosen such that $\Delta:=\det\tilde{W}$ be non-zero (notice that Δ does not depend on t due to the form of V and W). The change (17) transforms the residua as follows: $A_j^0\mapsto A_j^1=\tilde{W}(a_j)^{-1}A_j^0\tilde{W}(a_j)$, $j=1,\ldots,p+1$. At a_{p+2} a new singular point appears whose residuum is 0 (from the polar part only the term $O(1/(t-a_{p+2})^2)$ is non-zero).

7^0. The block U' of the residuum A_j^1 equals up to a non-zero factor $A_j|_{U_{\mu,\mu}}(V+\alpha_jW)|_{U'}-(V+\alpha_jW)|_{U'}A_j|_{U_{\nu,\nu}}$. This non-zero factor depends on δ. If $\gamma_\mu\neq\gamma_\nu$, then equation (16) shows that the blocks $A_j|_{U'}$ are obtained as a result of the change (17) in the fuchsian system with residua A_j^0.

This, however, is impossible because the sum of the residua of a meromorphic 1-form on $\mathbf{C}P^1$ is 0, so one should have $\sum_{j=1}^{p+1}A_j|_{U'}=0$; on the other hand, one has $\sum_{j=1}^{p+1}A_j|_{U'}=-A_{p+2}|_{U'}\neq 0$ by construction (see the proof of Lemma 32) – a contradiction. Hence, one has $\gamma_\mu=\gamma_\nu$ for all (μ,ν). As $V\in sl(n_s,\mathbf{C})$, one must have (for all i) $\gamma_i=0$. Thus only the entries of V belonging to the block F can be non-zero.

8^0. Sum up equations (16) from 1 to $p+1$. As $\sum_{j=1}^{p+1}A_j^0=0$ and $\sum\alpha_jA_j=-B$, one gets $[-A_{p+2},V]+[B,W]=0$. The form of the matrices A_{p+2} and V implies that $[-A_{p+2},V]=0$. The matrix B is upper-triangular and has distinct eigenvalues. This and the form of W implies $W=0$.

But then one has $[A_j,V]=0$ for all j. Hence, $A_j|_{U_{\mu,\mu}}V|_{U'}-V|_{U'}A_j|_{U_{\nu,\nu}}=0$. The $(p+1)$-tuples of matrices $A_j|_{U_{\mu,\mu}}$ and $A_j|_{U_{\nu,\nu}}$ defining non-equivalent irreducible representations, this implies $V=0$.

The lemma is proved.

5.4 Proof of Lemma 28

1^0. Block decompose the matrices from $gl(n,\mathbf{C})$, the diagonal blocks being square, of sizes n_s, $n_{s-1}-n_s$, $n_{s-2}-n_{s-1}$, …, $n-n_1$. Recall that we denote by $H_{\mu,\nu}$ the block in position (μ,ν) (it is of size $(n_{s-\mu+1}-n_{s-\mu+2})\times(n_{s-\nu+1}-n_{s-\nu+2})$; we set $n_{s+1}=0$). Denote by L_k the left upper block $n_{s-k+1}\times n_{s-k+1}$ (one has $L_1=H_{1,1}$).

2^0. Construct the matrices $A_j|_{L_1}$. For $j=1,\ldots,p+1$ one has $J(A_j|_{L_1})=J_j^{n_s}$, for $j=p+2$ the Jordan normal form of $A_{p+2}|_{L_1}$ consists of m_0 blocks of size 2 and of n_s-2m_0 ones of size 1; the sizes n_ν being divisible by $d>m_0$, one has $n_s\geq 2d>2m_0$ (one can't have $n_s=d$ because this would mean that the matrices $A_j|_{L_1}$ are scalar – a contradiction with condition (ω_{n_s})).

Lemma 31 *In Case A) there exists an irreducible (p+1)-tuple of matrices $A_j^s:=A_j|_{L_1}$ satisfying condition (14), with zero sum and with $J(A_j^s)=J_j^{n_s}$ for $j=1,\ldots,p+1$. The matrix $B^s:=\sum_{j=1}^{p+1}\alpha_jA_j^s$ is in Jordan normal form, its first d eigenvalues are simple and non-zero.*

The lemma results from Theorem 34, see Section 6.

Lemma 32 *In Case B) there exists a $(p+2)$-tuple of matrices $A_j^s:=A_j|_{L_1}$ satisfying condition (14), with zero sum, with trivial centralizer and with $J(A_j^s)=J_j^{n_s}$ for $j=1,\ldots,p+1$; $J(A_{p+2}^s)$ consists of m_0 blocks of size 2 and of n_s-2m_0 blocks of size 1 and the matrix algebra generated by A_1^s, $\ldots$, A_{p+2}^s contains a non-degenerate matrix. The matrix $B^s:=\sum_{j=1}^{p+1}\alpha_jA_j^s$ is upper-triangular and has distinct non-zero eigenvalues.*

The lemma is proved in the next subsection.

3^0. Suppose that the matrices $A_j^{s-k+1}:=A_j|_{L_k}$ (whose sum is 0) are constructed such that

a) for $j=1,\ldots,p+1$ one has $A_j^{s-k+1}\in J_j^{n_{s-k+1}}$ or A_j^{s-k+1} belongs to the closure of $J_j^{n_{s-k+1}}$;

b) in case B) $J(A_{p+2})$ consists of m_0 blocks of size 2 and of $n_{s-k+1}-2m_0$ blocks of size 1; in case A) $A_{p+2}=0$;

c) the matrices A_j^{s-k+1} are block upper-triangular, with $A_j^{s-k+1}|_{H_{\mu,\nu}}=0$ for $\mu>\nu$ and for $\mu=\nu>1$;

d) the columns of the $(p+1)$-tuple of matrices A_j^{s-k+1} $j\leq p+1$ are linearly independent.

The last conditions means that if $(A_j^{s-k+1})^i$ denotes the i-th column of the matrix A_j^{s-k+1} and if $\sum_{i=1}^{n_{s-k+1}}\beta_i(A_j^{s-k+1})^i=0$ for some constants $\beta_i\in\mathbf{C}$ and for $j=1,\ldots,p+1$, then $\beta_1=\ldots=\beta_{n_{s-k+1}}=0$.

Remark: The entries of the matrix A_{p+2} outside L_1 are 0. Therefore the construction from this moment on goes on like in [Ko1] and we omit some details.

4^0. Construct the matrices A_j^{s-k}. Set $A_j^{s-k}|_{H_{k+1,\nu}}=0$ for $\nu=1,\ldots,k+1$. Hence, condition c) will hold.

Define the linear subspaces $S_{j,k}\subset\mathbf{C}^{n_{s-k+1}}$ of vector-columns as follows. Fix matrices $Q_j\in GL(n_{s-k+1},\mathbf{C})$ such that the matrices $T_j:=Q_j^{-1}A_j^{s-k+1}Q_j$ be in upper-triangular Jordan normal forms. Then for $j=1,\ldots,p+1$ the space $S_{j,k}$ is spanned by the vectors of the form Q_jV where the non-zero coordinates of V can be only in the rows where T_j has units and in the last rows of the smallest $r(J_j^{n_{s-k}})-\mathrm{rk}(A_j^{s-k+1})$ Jordan blocks of T_j. (In some cases one has to choose part of the blocks of given size; there are no conditions imposed on the choice.)

For $j=1,\ldots,p+1$ one has $\dim(S_{j,k})=\mathrm{rk}(A_j^{s-k+1})+r(J_j^{n_{s-k}})-\mathrm{rk}(A_j^{s-k+1})=r(J_j^{n_{s-k}})$.

Denote by D_k the union of blocks $H_{1,k+1}\cup\ldots\cup H_{k,k+1}$.

Lemma 33 *Let for $j=1,\ldots,p+1$ the matrix A_j^{s-k+1} belong to the conjugacy class $J_j^{n_{s-k+1}}$ or to its closure. If the columns of $A_j^{s-k}|_{D_k}$ belong to the space $S_{j,k}$, then the matrix A_j^{s-k} belongs to the conjugacy class $J_j^{n_{s-k}}$ or to its closure.*

The reader can find the proof of this fact in [Ko1].

5^0. Denote by Γ_k the linear space of $(p+1)$-tuples of vector-columns $Q_jV_j\in\mathbf{C}^{n_{s-k+1}}$ with zero sum where $V_j\in S_{j,k}$. One has $\dim(\Gamma_k)\geq n_{s-k}$. Indeed,

$$\dim(\Gamma_k) \geq \dim(S_{1,k}) + \ldots + \dim(S_{p+1,k}) - n_{s-k+1} =$$

$$= r(J_1^{n_{s-k}}) + \ldots + r(J_{p+1}^{n_{s-k}}) - n_{s-k+1} = n_{s-k} + n_{s-k+1} - n_{s-k+1} = n_{s-k}$$

We admit that the n_{s-k+1} conditions arizing from $Q_1V_1 + \ldots + Q_{p+1}V_{p+1} = 0$ might not be linearly independent which explains why there is an inequality.

Define $\Delta_k \subset \mathbf{C}^{n_{s-k+1}}$ as the space spanned by the $(p+1)$-tuples of vector-columns of the matrices A_j^{s-k+1}, $j \leq p+1$. Hence, $\dim(\Delta_k) = n_{s-k+1}$ and $\dim(\Gamma_k/\Delta_k) \geq n_{s-k} - n_{s-k+1}$.

6^0. Hence, one can choose $n_{s-k} - n_{s-k+1}$ columns of the $(p+1)$-tuple of matrices $A_j^{s-k}|_{D_k}$ which belong to Γ_k and are linearly independent modulo the space Δ_k. With this choice the columns of the $(p+1)$-tuple of matrices A_j^{s-k} will be linearly independent, i.e. condition c) from 3^0 will hold. Condition b) holds by construction. Condition a) follows from Lemma 33. Thus conditions a) – d) hold for every k (and, in particular, for $k = s+1$).

7^0. Prove that if the matrices $A_j = A_j^0$ are constructed like this, then the centralizer $\mathcal{Z}'$ of their $(p+2)$-tuple is trivial. Let $X \in \mathcal{Z}'$. Denote by F a matrix from the algebra generated by the matrices A_j whose restriction F' to L_1 is non-degenerate. Such a matrix exists – in Case A) this follows from the irreducibility of the algebra generated by the matrices A_j^s, in Case B) this follows from Lemma 32.

The commutation relation $[X,F]|_{H_{s+1,1}} = 0$ implies $X|_{H_{s+1,1}} = 0$ (because $F|_{H_{\mu,\nu}} = 0$ if $\mu > \nu$ and if $\mu = \nu > 1$, and one has $[X,F]|_{H_{s+1,1}} = (XF)|_{H_{s+1,1}} = X|_{H_{s+1,1}}F' = 0$ and F' is non-degenerate).

The commutation relation $[X,F]|_{H_{s,1}} = 0$ implies $X|_{H_{s,1}} = 0$ (because $X|_{H_{s+1,1}} = 0$ and one has $[X,F]|_{H_{s,1}} = (XF)|_{H_{s,1}} = X|_{H_{s,1}}F' = 0$). Similarly $X|_{H_{k,1}} = 0$ for $k > 1$.

For $k = 1$ the commutation relations $[X, A_j]|_{H_{1,1}} = 0$ are equivalent to $[X|_{H_{1,1}}, A_j^s] = 0$, hence, $X|_{H_{1,1}} = \alpha I$ by triviality of the centralizer of the $(p+2)$-tuple of matrices A_j^s, see Lemmas 31 and 32. Assume that $\alpha = 0$.

8^0. After 7^0 the commutation relations $[X, A_j]|_{H_{k,2}} = 0$ $(k = 1, \ldots, s+1)$ become equivalent to $(A_jX)|_{H_{k,2}} = 0$. They imply $X|_{H_{k,2}} = 0$, otherwise the columns of the $(p+2)$-tuple of matrices A_j would be linearly dependent – a contradiction with condition d). In the same way one deduces that $X|_{H_{k,l}} = 0$ for $k = 1, \ldots, s+1$; $l \geq 2$. Hence, $X = 0$. Without the assumption $\alpha = 0$, see 7^0, this means that $X = \alpha I$, i.e. $\mathcal{Z}'$ is trivial.

Conditions 1) – 8) of the lemma follow immediately from the construction of the matrices A_j. The lemma is proved.

5.5 Proof of Lemma 32

1^0. Block-decompose the matrices from $gl(n_s, \mathbf{C})$, all diagonal blocks being of size $\chi =$ 2,3,4 or 6 (recall that the Jordan normal forms $J_j^{n_s}$ correspond to one of the *special cases* from Subsection 6.1). Denote the blocks by $U_{\mu,\nu}$. Construct a block-diagonal $(p+1)$-tuple of matrices A_j^0 with diagonal blocks defining irreducible non-equivalent representations. If they are constructed like in examples (ex0) – (ex7) from Subsection 6.2, see Lemma 39 as well, then the matrix B can have distinct non-zero eigenvalues (recall that one has the right to multiply the diagonal blocks by non-zero constants – this preserves the conjugacy classes of A_j and changes the eigenvalues of B).

2^0. Define A_{p+2}. Denote by F' the right upper block $m_0 \times (n_s - m_0)$ (recall that $m_0 < d \leq n_s/2$). If m_0 is divisible by the size χ, then set $F = F'$ and $n' = n_s - m_0$. If not, then define

F'' as obtained from F' by deleting its first (i.e. left) $\chi - \psi$ columns where ψ is the rest of the division of m_0 by χ; set $n' = n_s - m_0 - \chi + \psi$.

Notice that the block F'' is a union of $[m_0/\chi]$ rows each of $[n'/\chi]$ entire blocks $U_{\mu,\nu}$ and (when $\psi > 0$) of a row of $[n'/\chi]$ non-entire blocks $U_{\mu,\nu}$ (of which only the first ψ rows belong to F''). Denote by F the block which is the union of all blocks $U_{\mu,\nu}$ which entirely or partially belong to F''. Denote by ${}^tF$ the transposed to F.

Require the restriction of A_{p+2} to F'' to be of maximal rank and its restriction to every block $U_{\mu,\nu}$ which entirely or partially belongs to F'' to be non-zero. Hence, $J(A_{p+2})$ consists of m_0 blocks of size 2 and of $n - 2m_0$ ones of size 1.

3^0. Define the matrices A_j, $j \leq p+1$. Their restriction to every diagonal block $U_{\mu,\mu}$ equals the one of A_j^0 to it, their restriction to every block $U_{\mu,\nu} \subset F$ is of the form $A_j^0|_{U_{\mu,\mu}} D_{j;\mu,\nu} - D_{j;\mu,\nu} A_j^0|_{U_{\nu,\nu}}$, $D_{j;\mu,\nu} \in gl(\chi, \mathbf{C})$, their other blocks are 0.

The representations defined by $A_j^0|_{U_{\mu,\mu}}$ and $A_j^0|_{U_{\nu,\nu}}$ being non-equivalent for $\mu \neq \nu$, the map

$$(D_1, \ldots, D_{p+1}) \mapsto \sum_{j=1}^{p+1} (A_j^0|_{U_{\mu,\mu}} D_j - D_j A_j^0|_{U_{\nu,\nu}})$$

is surjective onto $gl(\chi, \mathbf{C})$. Hence, one can choose the blocks $D_{j;\mu,\nu}$ such that the sum of the matrices A_j to be 0.

Notice that for all j the matrix A_j is conjugate to A_j^0.

4^0. There remains to be proved that the centralizer $\mathcal{Z}$ of the $(p+1)$-tuple of matrices A_j is trivial. Denote by U a block U_{μ_0,ν_0} from ${}^tF$. For $X \in \mathcal{Z}$ the commutation relation $[A_{p+2}, X] = 0$ restricted to U_{μ_0,ν_0} yields $X_U = 0$.

One has $A_j|_{U_{\mu_0,\mu_0}} X|_{U_{\mu_0,i}} - X|_{U_{\mu_0,i}} A_j|_{U_{i,i}} = 0$, hence, if $i \neq \mu_0$, then $X|_{U_{\mu_0,i}} = 0$, if $i = \mu_0$, then $X|_{U_{\mu_0,i}} = \gamma_i I$. In the same way one deduces that $X|_{U_{i,\nu_0}} = 0$, if $i \neq \nu_0$, and $X|_{U_{i,\nu_0}} = \gamma_i I$ if $i = \nu_0$.

Let $U' = U_{\mu_1,\nu_1} \subset F$. Set $X_{U'} = X'$, $A_j|_{U'} = A'_j$ and recall that $X_{U_{i,i}} = \gamma_i I$. The commutation relations restricted to U' yield

$$(\gamma_{\mu_1,\mu_1} - \gamma_{\nu_1,\nu_1}) A'_j + A_j|_{U_{\mu_1,\mu_1}} X' - X' A_j|_{U_{\nu_1,\nu_1}} = 0$$

Sum them up from 1 to $p+1$. One has $\sum_{j=1}^{p+1} A_j|_{U_{i,i}} = 0$ for all i. Hence, the sum is $(\gamma_{\mu_1,\mu_1} - \gamma_{\nu_1,\nu_1}) A_{p+2}|_{U'}$ which is 0 only if $\gamma_{\mu_1,\mu_1} = \gamma_{\nu_1,\nu_1}$. But then one must have $A_j|_{U_{\mu_1,\mu_1}} X' - X' A_j|_{U_{\nu_1,\nu_1}} = 0$ for all j. The non-equivalence of the representations defined by the blocks $A_j|_{U_{i,i}}$ for the different values of i implies that $X' = 0$. Hence, the centralizer is trivial.

The lemma is proved.

6 On the existence of irreducible representations defined by nilpotent matrices

6.1 The basic result

In this section we consider the question of existence of irreducible $(p+1)$-tuples of nilpotent matrices from given conjugacy classes c_j, with given quantities $r_j = r(c_j)$ and with zero sum. This question was considered in [Ko2]. We have to repeat some of the reasoning from [Ko2] for two reasons:

– there is an inexactitude in the formulation of the basic result from [Ko2], so we explain here what is correct and what is wrong;

– we want to prove something more than the existence of such $(p+1)$-tuples.

Define as *special* the following cases. In each case the Jordan normal form defined by the class c_j has Jordan blocks of one and the same size l_j. The special cases are

$$\begin{array}{ll} a) & p=3, \quad n=2g, \quad g>1, \quad l_1=l_2=l_3=l_4=2; \\ b) & p=2, \quad n=3g, \quad g>1, \quad l_1=l_2=l_3=3; \\ c) & p=2, \quad n=4g, \quad g>1, \quad l_1=l_2=4, \ l_3=2; \\ d) & p=2, \quad n=6g, \quad g>1, \quad l_1=6, \ l_2=3, \ l_3=2 \end{array}$$

Define also some *almost special* cases. They are obtained from one of the special ones (it is understood from the notation from which) by replacing in one of the Jordan normal forms a couple of blocks of size l_j by two blocks of sizes l_j+1 and l_j-1. For these cases we give the sizes of the blocks:

$$\begin{array}{lllll} a1) & J_1 : 3,1,2,\ldots,2 & J_2,J_3,J_4 : 2,\ldots,2 & & n=2g, \ g>1 \\ b1) & J_1 : 4,2,3,\ldots,3 & J_2 : 3,\ldots,3 & J_3 : 3,\ldots,3 & n=3g, \ g>1 \\ c1) & J_1 : 4,\ldots,4 & J_2 : 4,\ldots,4 & J_3 : 3,1,2,\ldots,2 & n=4g, \ g>1 \\ c2) & J_1 : 5,3,4,\ldots,4 & J_2 : 4,\ldots,4 & J_3 : 2,\ldots,2 & n=4g, \ g>1 \\ d1) & J_1 : 6,\ldots,6 & J_2 : 3,\ldots,3 & J_3 : 3,1,2,\ldots,2 & n=6g, \ g>1 \\ d2) & J_1 : 6,\ldots,6 & J_2 : 4,2,3,\ldots,3 & J_3 : 2,\ldots,2 & n=6g, \ g>1 \\ d3) & J_1 : 7,5,6,\ldots,6 & J_2 : 3,\ldots,3 & J_3 : 2,\ldots,2 & n=6g, \ g>1 \end{array}$$

Definition. A representation defined by a $(p+1)$-tuple of matrices A_j with zero sum is called *nice* if it is either irreducible or is reducible, with a trivial centralizer and the matrices A_j admit a simultaneous conjugation to a block upper-triangular form in which the restrictions of the $(p+1)$-tuple to all diagonal blocks (they are all of sizes >1) define non-equivalent irreducible representations. (Hence, the algebra defined by the matrices A_j contains a non-degenerate matrix.)

We prove in this section the following :

Theorem 34 *1) Let the $(p+1)$-tuple of nilpotent conjugacy classes $c_j \in gl(n,\mathbf{C})$ be given with $r_1+\ldots+r_{p+1} \geq 2n$ and not corresponding to any of the special cases. Then there exists a $(p+1)$-tuple of matrices $A_j \in c_j$ defining a nice representation and there exists a $(p+1)$-tuple of distinct non-zero complex numbers α_j such that the matrix $B := \alpha_1 A_1 + \ldots + \alpha_{p+1}A_{p+1}$ has a simple non-zero eigenvalue.*

2) In the conditions of 1), if the $(p+1)$-tuple of nilpotent conjugacy classes $c_j \in gl(n,\mathbf{C})$ does not correspond to any of the almost special cases a1), b1), c2), or d3), then there exists an irreducible $(p+1)$-tuple of matrices $A_j \in c_j$ satisfying (2). In all 7 almost special cases one can obtain a matrix B with distinct non-zero eigenvalues.

3) If the conjugacy classes C_j are unipotent, satisfying $r_1+\ldots+r_{p+1} \geq 2n$ and not corresponding to any of the special cases, then there exists a $(p+1)$-tuple of matrices $M_j \in C_j$ defining a nice representation and satisfying (1). If they do not correspond to any of the almost special cases a1), b1), c2), or d3), then there exists an irreducible $(p+1)$-tuple of matrices $M_j \in C_j$ satisfying (1).

Remarks: 1) Condition (ω_n) is necessary for the existence of nice representations – it must hold for the restrictions of the matrices to each of the diagonal blocks and the rank of a nilpotent matrix is $\geq$ the sum of the ranks of its restrictions to these blocks.

2) In [Ko2] the existence of irreducible representations in cases a1), b1), c2) and d3) is claimed which is not true. The proof of the basic result from [Ko1] (which uses the result

from [Ko2]) needs only the existence of nice representations (not necessarily irreducible) and is performed with the corrected result in exactly the same way as it is done in [Ko1]. The rest of the results from [Ko2] are correct.

The theorem is proved in Subsection 6.3. We don't prove part 3) of it which follows from parts 1) and 2) when one considers as matrices M_j the monodromy operators of a fuchsian system with matrices-residua A_j satisfying the conditions of part 1) or 2) (remember Proposition 11). Condition (ω_n) is necessary for the existence of irreducible $(p+1)$-tuples of nilpotent matrices satisfying (2) or of unipotent matrices M_j satisfying (1), see [Ko2].

We precede the proof of the lemma by the construction of examples of irreducible triples or quadruples of matrices A_j in some particular cases, see the next subsection, and by deducing Corollary 35 from the theorem. We refer to the examples as to (ex1), (ex2) etc. The irreducibility of the triples or quadruples from each example is checked by proving that the matrix algebra generated by the matrices is $gl(n,\mathbf{C})$. To this end one first finds a matrix S from the algebra with a single non-zero entry (by multiplying in a suitable order the matrices) and then again by suitable multiplications of the matrix S by the matrices A_j one obtains all other elements of the canonical basis of $gl(n,\mathbf{C})$, see examples of applications of this technique in [Ko2].

Corollary 35 *Let $d>1$ and let the $(p+1)$-tuple of nilpotent conjugacy classes $c_j\in gl(n,\mathbf{C})$ be given with $r_1+\ldots+r_{p+1}\geq 2n$ and not corresponding to any of the special cases. Then there exists an irreducible $(p+1)$-tuple of matrices $A_j\in c_j$ satisfying (2) and a $(p+1)$-tuple of distinct non-zero complex numbers α_j such that the matrix $B:=\alpha_1A_1+\ldots+\alpha_{p+1}A_{p+1}$ has at least d simple non-zero eigenvalues.*

Proof:

1^0. Denote by $c_j^1\in gl(n/d,\mathbf{C})$ the conjugacy classes obtained from c_j by reducing the size of the matrices and the number of Jordan blocks of a given size d times. Denote by c_j^l the conjugacy classes obtained from c_j^1 by increasing the size of the matrices and the number of Jordan blocks of a given size l times. In particular, $c_j^d=c_j$.

2^0. Suppose first that the conjugacy classes c_j^1 don't correspond to an almost special case (they don't correspond to a special case, even with $g=1$, because in this case c_j would also correspond to such a case). We show that there exist irreducible $(p+1)$-tuples of matrices $A_j^l\in c_j^l$. For $l=1$ this follows from the above theorem. Suppose that it is true for $l\leq l_0$. Then there exists an irreducible triple of matrices $A_j^{l_0}\in c_j^{l_0}$ whose sum is zero.

3^0. Consider the matrices $\left(\begin{array}{cc}A_j^{l_0}&0\\0&A_j^1\end{array}\right)$. The irreducible representations L^{l_0} and L^1 defined by the triples of matrices $A_j^{l_0}$ and A_j^1 satisfy the condition $\dim\ \mathrm{Ext}^1(L^{l_0},L^1)\geq 2$. Indeed, one has $\dim\ \mathrm{Ext}^1(L^{l_0},L^1)=l_0(\sum_{j=1}^{p+1}d(c_j^1)-2(n/d)^2)\geq 2$ because $\sum_{j=1}^{p+1}d(c_j^1)\geq 2n^2+2$ (see [Ko2], Lemma 3). One subtracts $(n/d)^2$ twice – once to factor out conjugations with block upper-triangular matrices and once because the sum of the matrices is 0.

Lemma 36 *1) If for the non-equivalent irreducible representations L', L'' defined by the matrices $A_j'\in c_j'$, $A_j''\in c_j''$ (each satisfying (2)) one has dim $Ext^1(L',L'')\geq 1$, then there exists a representation defined by matrices A_j''' satisfying (2), with $A_j'''\in c_j'\oplus c_j''$, which is their semi-direct sum. If dim $Ext^1(L',L'')\geq 2$, then there exists an irreducible such representation. The conjugacy classes c_j', c_j'' are arbitrary (not necessarily nilpotent).*

2) Let the conjugacy classes c_j', c_j'' be nilpotent. Let the matrices $B^{(i)}=\sum_{j=1}^{p+1}\alpha_jA_j^{(i)}$, $i=1,2$ have respectively m' and m'' simple non-zero eigenvalues. If the matrices $A_j^{(i)}$ are like in 1) and

if dim $Ext^1(L',L'')\geq 2$*, then there exists an irreducible representation defined by matrices* A_j *satisfying (2), with* $A_j\in c_j'\oplus c_j''$ *and such that the matrix* B *(defined like* B'*,* B''*) has at least* $m'+m''$ *simple non-zero eigenvalues.*

The lemma is proved after the corollary. It implies the existence of irreducible triples of matrices $A_j^{l_0+1}\in c_j^{l_0+1}$ with at least l_0+1 simple non-zero eigenvalues.

4^0. If the conjugacy classes c_j^1 correspond to an almost special case, then for $l>1$ the classes c_j^l correspond to a $(p+1)$-tuple of conjugacy classes which contains in its closure a *neighbouring* case (their definition is given in **8.** of Subsubsection 6.3.1). For such cases we prove that there exist irreducible $(p+1)$-tuples with B having distinct non-zero eigenvalues, see Subsubsection 6.3.7. Hence, one can choose an irreducible $(p+1)$-tuple of matrices $A_j\in c_j^l$ which is close to a neighbouring case and B will still be with distinct non-zero eigenvalues.

The corollary is proved.

Proof of Lemma 36:

1^0. As dim $\mathrm{Ext}^1(L',L'')\geq 1$, there exists a semi-direct sum of L' and L'' which is not reduced to a direct one, i.e. there exists a $(p+1)$-tuple of matrices $A_j'''=\left(\begin{array}{cc}A_j'&G_j\\0&A_j''\end{array}\right)$ which by simultaneous conjugation can't be transformed into a block-diagonal one.

2^0. The representations L', L'' are not equivalent which means that the centralizer $\mathcal{Z}$ of the $(p+1)$-tuple of matrices A_j''' is trivial (if $\left(\begin{array}{cc}U&M\\N&P\end{array}\right)\in\mathcal{Z}$, then $NA_j'-A_j''N=0$; as L' and L'' are irreducible and non-equivalent, one has $N=0$; then $[A_j',U]=0$ and $[A_j'',P]=0$, i.e. $U=\alpha I$, $P=\beta I$ (Schur's lemma); then $A_j'M-MA_j''=(\beta-\alpha)G_j$ which means that $M=0$, $\alpha=\beta$, otherwise the sum of L' and L'' must be direct).

3^0. The condition dim $\mathrm{Ext}^1(L',L'')\geq 2$ implies the existence of infinitesimal conjugations of A_j''' by matrices $I+\varepsilon X_j$, $X_j=\left(\begin{array}{cc}Y_j&0\\Z_j&T_j\end{array}\right)$ which are not tantamount to a simultaneous infinitesimal conjugation of the $(p+1)$-tuple of matrices A_j'''. Indeed, one has

$$\tilde{A}_j:=(I+\varepsilon X_j)^{-1}A_j'''(I+\varepsilon X_j)=A_j'''+\varepsilon\left(\begin{array}{cc}[A_j',Y_j]+G_jZ_j&0\\A_j''Z_j-Z_jA_j'&[A_j'',T_j]-Z_jG_j\end{array}\right)+o(\varepsilon)$$

Set $\Phi=\{(P_1,\ldots,P_{p+1})|P_j=A_j''Z_j-Z_jA_j',\sum_{j=1}^{p+1}P_j=0\}$, $\Psi=\{(Q_1,\ldots,Q_3)|Q_j=A_j''Z-ZA_j'\}$ (as $\sum_{j=1}^{p+1}A_j^{(i)}=0$, $i=1,2$, one has $\sum_{j=1}^{p+1}Q_j=0$). The condition dim $\mathrm{Ext}^1(L',L'')\geq 2$ implies that $\dim\Phi-\dim\Psi\geq 2$.

4^0. Denote by Ξ the codimension 1 subspace of Φ satisfying the condition $\mathrm{tr}(\sum_{j=1}^{p+1}G_jZ_j)=0$. Hence, if the $(p+1)$-tuple of blocks Z_j of the matrices X_j belongs to Ξ/Ψ, then one can find blocks Y_j and T_j such that $\sum_{j=1}^{p+1}\tilde{A}_j=o(\varepsilon)$ (i.e. one can solve the equations $\sum_{j=1}^{p+1}[A_j',Y_j]+G_jZ_j=0$ and $\sum_{j=1}^{p+1}([A_j'',T_j]-Z_jG_j)=0$ w.r.t. Y_j and T_j).

5^0. Hence, one can find a conjugation of A_j''' by matrices $I+\varepsilon X_j+\varepsilon^2S_j(\varepsilon)$ analytic in ε, with X_j as above and such that the sum of the conjugated matrices (denoted by A_j) to be 0 (identically in ε). The existence of S_j is proved by complete analogy with the basic technical tool.

As $(X_1,\ldots,X_{p+1})\in(\Xi/\Psi)$, the $(p+1)$-tuple is irreducible. (Indeed, the matrix algebra $\mathcal{A}$ generated by the matrices $\tilde{A}_j$ contains matrices of the form $Y'=Y+O(\varepsilon)$ for every matrix Y blocked as A_j'''; this follows from the main result of [Ko4]. In particular, for $Y=H=\left(\begin{array}{cc}I&0\\0&0\end{array}\right)$.

Conjugate Y' (with $Y = H$) by a matrix $I + O(\varepsilon)$ to make its left lower block 0 (identically in ε). This conjugation can't annihilate the left lower blocks of all matrices from $\mathcal{A}$ due to $(X_1, \ldots, X_{p+1}) \in (\Xi/\Psi)$. If $S \in \mathcal{A}$ has a non-zero such block, then $SH - HSH$ has the same left lower block and all its entries are $O(\varepsilon)$. Multiplying S by matrices of the form Y' and adding the matrices Y', one obtains a basis of $gl(n, \mathbf{C})$.)

This proves 1).

6^0. Prove 2) One can assume that the matrices B', B'' have no non-zero eigenvalue in common. This can be achieved by multiplying one of the $(p+1)$-tuples of matrices $A_j^{(i)}$ by $h \in \mathbf{C}^*$ (we use here the fact that the matrices are nilpotent – such a multiplication does not change the conjugacy classes). Hence, the matrix B''' has at least $m' + m''$ simple non-zero eigenvalues. For $\varepsilon \neq 0$ small enough the eigenvalues of the matrix $B = \alpha_1 A_1 + \ldots + \alpha_{p+1} A_{p+1}$ will be close to the ones of B''', hence, it will have at least $m' + m''$ non-zero simple eigenvalues.

The lemma is proved.

6.2 Some examples

We explain two methods for constructing irreducible triples or quadruples of nilpotent matrices. The first one places the non-zero entries of the matrices A_j in positions $(k, k+1)$, $k = 1, \ldots, n-1$, and $(n, 1)$. In all examples except (ex0) one has $p = 2$.

Example (ex0): Let $n = 2$, $p = 3$. Set $A_1 = -A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $A_3 = -A_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. The quadruple is irreducible and for almost all values of α_j the matrix B has two different non-zero eigenvalues.

Example (ex1): Let $n \geq 4$. Let

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \ldots & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \end{pmatrix},$$

$A_3 = -A_1 - A_2$. One checks directly that the matrices are nilpotent, that $r_1 = r_2 = n - 1$, $r_3 = 2$ and $(A_3)^2 = 0$.

Example (ex2): Let $n = 3$. Let $A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}$. Hence, each matrix is nilpotent, of rank 2.

Lemma 37 *The characteristic polynomial of a matrix having non-zero entries in positions $(k, k+1)$, $k = 1, \ldots, n-1$, and $(n, 1)$ and zeros elsewhere is of the form $\lambda^n + a$, $a \neq 0$. Hence, the eigenvalues of such a matrix are all non-zero and distinct.*

The lemma is to be checked directly. It implies that the eigenvalues of the matrices B defined after examples (ex1) and (ex2) are non-zero and distinct.

Another method is the non-zero entries to be in positions $(k, k+1)$, $k = 1, \ldots, n-1$, and $(n-1, 1)$, $(n, 2)$. The idea to give such examples – the first of the matrices is initially in Jordan normal form and then one conjugates it with $I + E_{n,1}$. As our matrices will be sparce, we'll list only the entries of these positions, in this order. E.g., we write $A_1 \ : \ 1\ 1\ 0 \mid 1\ \ -1$ instead of $A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$ (the vertical line separates the last two entries just for convenience).

The reader is advised to draw in each example the matrices oneself.

Example (ex3): Let $n = 6$. Let

$$\begin{array}{lcrrrrrcrr} A_1 & : & 1 & 1 & 1 & 1 & 1 & | & 1 & -1 \\ A_2 & : & -1 & -1 & 0 & -1 & -1 & | & 0 & 0 \\ A_3 & : & 0 & 0 & -1 & 0 & 0 & | & -1 & 1 \end{array}$$

Hence, $J(A_1)$ consists of a single block of size 6, $J(A_2)$ consists of two blocks of size 3 and $J(A_3)$ consists of three blocks of size 2. One can give examples when n is even, $J(A_2)$ consists of two blocks of size 3 and of $(n-6)/2$ blocks of size 2; $J(A_3)$ consists of $n/2$ blocks of size 2; $J(A_1)$ consists of a single block of size n. To this end one adds to the right of the fourth from the left column of numbers in the above example a pack of $n-6$ units in the row of A_1, a pack of $(n-6)/2$ groups of the form $0, -1$ in the row of A_2 and a pack of $(n-6)/2$ groups of the form $-1, 0$ in the row of A_3.

Example (ex4): Let $n = 9$. Let

$$\begin{array}{lcrrrrrrrrcrr} A_1 & : & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & | & 1 & -1 \\ A_2 & : & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & | & 0 & 0 \\ A_3 & : & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 0 & | & -1 & 1 \end{array}$$

One checks directly that $J(A_1)$ consists of two Jordan blocks, of sizes 4 and 5, $J(A_2)$ consists of three Jordan blocks of size 3 and $J(A_3)$ consists of three Jordan blocks of size 2 and of one of size 3.

Example (ex5): Let $n = 10$. Let

$$\begin{array}{lcrrrrrrrrrcrr} A_1 & : & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & | & 1 & -1 \\ A_2 & : & -1 & -1 & 0 & -1 & -1 & -1 & 0 & -1 & -1 & | & 0 & 0 \\ A_3 & : & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & | & -1 & 1 \end{array}$$

In this example $J(A_1)$ consists of two Jordan blocks of size 5, $J(A_2)$ consists of two Jordan blocks of size 3 and of one of size 4, $J(A_3)$ consists of five Jordan blocks of size 2.

Example (ex6): Let $n = 12$. Let

$$\begin{array}{lcrrrrrrrrrrrcrr} A_1 & : & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & | & 1 & -1 \\ A_2 & : & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & | & 0 & 0 \\ A_3 & : & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & | & -1 & 1 \end{array}$$

Hence, $J(A_1)$ consists of three Jordan blocks of size 4, $J(A_2)$ consists of four Jordan blocks of size 3 and $J(A_3)$ consists of two Jordan blocks of size 3 and of three of size 2.

Example (ex7): Let $n = 5$. Let

$$\begin{array}{lcrrrrcrr} A_1 & : & 1 & 1 & 1 & 1 & | & 1 & -1 \\ A_2 & : & -1 & -1 & 0 & -1 & | & 0 & 0 \\ A_3 & : & 0 & 0 & -1 & 0 & | & -1 & 1 \end{array}$$

$J(A_1)$ consists of a single block of size 5, $J(A_2)$ and $J(A_3)$ consist each of a block of size 3 and of a block of size 2. One can give examples when $n>5$ is odd, each of $J(A_2)$ and $J(A_3)$ consists of one block of size 3 and of $(n-3)/2$ blocks of size 2; $J(A_1)$ consists of a single block of size n. To this end one adds to the left of the vertical line in the above example a pack of $n-5$ units in the row of A_1, a pack of $(n-5)/2$ groups of the form $0,-1$ in the row of A_2 and a pack of $(n-5)/2$ groups of the form $-1,0$ in the row of A_3.

Lemma 38 *The characteristic polynomial of a matrix having non-zero entries only in positions $(k,k+1)$, $k=1,\ldots,n-1$, and $(n-1,1)$, $(n,2)$ is of the form $\lambda^n+b\lambda$. Hence, when $b\neq 0$ its roots are all distinct and one of them equals 0.*

The lemma is to be checked directly.

Lemma 39 *For the triples of matrices A_j from each of the examples (ex3) – (ex7) there exist conjugations of the matrices A_j with matrices $I+O(\varepsilon)$ analytic in $\varepsilon\in(\mathbf{C},0)$ such that for $\varepsilon\neq 0$ the eigenvalues of the matrix B are distinct and non-zero.*

Proof:

1^0. In examples (ex4) – (ex6) an infinitesimal conjugation of A_1 with $I+\varepsilon(E_{2,1}-E_{n,n-1})$, of A_2 with $I+\varepsilon E_{2,1}$ and of A_3 with $I-\varepsilon E_{n,n-1}$ creates a new triple (of matrices A_j^1) satisfying (2) in first approximation w.r.t. ε. Hence, there exist true conjugations differing from the above ones by terms $O(\varepsilon^2)$ after which for the deformed matrices A_j^2 (2) holds. One has $A_j^1-A_j^2=o(\varepsilon)$.

2^0. Choose α_j such that the entries of $B^1|_{\varepsilon=0}$ in positions $(k,k+1)$, $k=1,\ldots,n-1$, and $(n-1,1)$, $(n,2)$ to be $\neq 0$. The entries in the following positions (and only they) of the matrix B^1 (hence, B^2 as well) are non-zero and $O(\varepsilon)$: $(1,1)$, $(2,2)$, $(n-1,n-1)$, (n,n) and $(n,1)$. Hence, $\det B^1=v\varepsilon+o(\varepsilon)$, $v\neq 0$ and for $\varepsilon\neq 0$ small enough all eigenvalues of B^1 and B^2 are non-zero and distinct, the last of them is $O(\varepsilon)$.

3^0. In examples (ex7) and (ex3) the triple of matrices A_j^1 is created by an infinitesimal conjugation of A_1 and A_2 with $I-\varepsilon E_{n,n-1}$ and of A_3 with $I+\varepsilon E_{2,1}$. The entries in the following positions (and only they) of the matrix B^1 (hence, B^2 as well) are non-zero and $O(\varepsilon)$: $(n-1,n-1)$, (n,n) and $(n,1)$. For the rest the reasoning is the same.

The lemma is proved.

Lemma 40 *1) Denote by Φ_1, Φ_2 two irreducible representations defined by any of examples (ex1) – (ex7) (with $n=6$ in (ex3) and $n=5$ in (ex7)). Then $Ext^1(\Phi_1,\Phi_2)\geq 2$.*

2) The same is true if each of Φ_1, Φ_2 corresponds to one of the examples (ex3) or (ex7) without the restrictions respectively $n=6$ and $n=5$.

The proof is left for the reader.

6.3 Proof of Theorem 34

6.3.1 Simplification and plan of the proof

1. The first observation to be made is that if there exists a $(p+1)$-tuple of matrices $A_j\in c_j$ satisfying 1) or 2) of the theorem, then there exist such $(p+1)$-tuples for every $(p+1)$-tuple of conjugacy classes c_j' where for each j either $c_j=c_j'$ or c_j is subordinate to c_j'. To this end one has to apply the basic technical tool.

2. Call *operation* (s,l), $s\geq l\geq 1$ the changing of a given nilpotent conjugacy class c containing two Jordan blocks of sizes s and l to c' in which these blocks are replaced by two

blocks of sizes $s+1$ and $l-1$. The class c lies in the closure of c'. If the nilpotent conjugacy class c lies in the closure of the nilpotent conjugacy class c'', then there exists a sequence of conjugacy classes $c_0 = c$, c_1, ... ,$c_\mu = c''$ such that c_i is obtained from c_{i-1} as a result of some operation (s,l). This is proved in [Ko1].

3. Let $A \in c$, $A' \in c'$, c, c' being nilpotent conjugacy classes. Then c lies in the closure of c' if and only if for all i one has $\mathrm{rk}(A)^i \leq \mathrm{rk}(A')^i$ (proved in [Ko1]).

4. For given $r \in \mathbf{N}$ denote by $\Omega_0(r)$ the nilpotent orbit of rank r of least dimension. It is unique and has Jordan blocks either of one and the same size or of two consecutive sizes. This follows from **2**. If the size is one and the same, then there exists a single orbit $\Omega_1(r)$ containing $\Omega_0(r)$ in its closure and contained in the closure of any orbit with the same value of r and different from $\Omega_0(r)$ and $\Omega_1(r)$. It is obtained from $\Omega_0(r)$ by an operation (k,k), k being the size of the blocks of $\Omega_0(r)$.

5. Let $r_1 + \ldots + r_{p+1} > 2n$. Then it is possible to change some of the conjugacy classes to subordinate ones of smaller rank to get the condition $r_1 + \ldots + r_{p+1} = 2n$ and the new $(p+1)$-tuple of ranks not to correspond to any of the ones of special or almost special cases.

6. Call *merging* of two nilpotent conjugacy classes c and c' the following procedure defined for $r(c) + r(c') \leq n-1$ and when $c = \Omega_0(r(c))$, $c' = \Omega_0(r(c'))$ (hence, at least one of the two Jordan normal forms has only blocks of size ≤ 2). If $A \in c$, $A' \in c'$ are Jordan matrices with decreasing order of the sizes b_i, b_i' of the blocks, $\mathrm{rk}A \geq \mathrm{rk}A'$ (hence, $b_i' \leq 2$), then construct the nilpotent matrix A'' as follows: insert on the first superdiagonal (it comprises the positions $(k,k+1)$) between the packs of $b_i - 1$ and $b_{i+1} - 1$ units from A a unit from A' as long as this is possible. These packs are units from Jordan blocks of sizes b_i and b_{i+1}. When inserting a unit, it takes the place of the 0 and the units of A do not change their positions.

Given the matrix A'' it is self-evident how to represent it in the form $A + A'$ with $A \in c$, $A' \in c'$. If there exists an irreducible or nice representation in which one of the matrices is from c'', then there exists such a representation with one more matrix, two of the matrices being from c and c', the other conjugacy classes remaining the same.

7. We prove the theorem only in the case $r_1 + \ldots + r_{p+1} = 2n$ making use of **5.** Making use of **4.**, when $p \geq 4$, one might restrict oneself to the case when $c_j = \Omega_0(r_j)$ for all j.

It is also possible to restrict oneself to the cases $p = 2$ and $p = 3$ due to the possibility to merge Jordan normal forms. Indeed, if $r_1 + \ldots + r_{p+1} = 2n$, if $c_j = \Omega_0(r(c_j))$ for all j and if $p \geq 4$, then a merging is always possible. Moreover, it is possible to be done with avoiding to come to the special case a) or to the almost special case a1). Indeed, when passing from $p = 4$ to $p = 3$ by a merging, the sum of three of the quantities r_j is $\leq n$ and one can choose a couple to be merged such that one of the four quantities r_j which remain after the merging to be $< n/2$.

When passing from $p = 3$ to $p = 2$ by merging (the passage is not defined for the cases a) and a1)), one can avoid to come to any of the other special or almost special cases. Indeed, merging results either in a Jordan normal form with a single Jordan block of size $m > 1$ or with greatest difference h between the sizes of two of the Jordan blocks ≥ 2; one has $m \geq 3$ and the cases $m = 3$ and $h = 2$ are possible only if one of the Jordan normal forms to be merged has only one block of size 2, the rest of size 1. In this case one can merge other two of the Jordan normal forms to avoid the special and almost special cases.

8. Finally, after having restricted oneself to the cases $p = 2$ and $p = 3$, one can again use **1.**, **2.** and **4.** to replace the given conjugacy classes by the triple or quadruple of conjugacy classes $\Omega_0(r_j)$. This is not always possible because one could come to a special or almost special case.

Let this be not a special or an almost special case. Use the notation "$A_1 = 2$" or "$A_1 = (2,3)$" in the sense "the Jordan normal form of A_1 has only Jordan blocks of size 2" ("of sizes 2 and 3"). The following cases of triples $\Omega_0(r_1)$, $\Omega_0(r_2)$, $\Omega_0(r_3)$ are possible (see [Ko2]):

(A)	$A_1=(1,2)$	(B)	$A_1=2$ or $A_1=(2,3); A_2=(2,3)$
(C)	$A_1=(2,3); A_2=3; A_3=(3,4)$	(D)	$A_1=(2,3); A_2=3; A_3=4$
(E)	$A_1=(2,3); A_2=3; A_3=(4,5)$	(F)	$A_1=(2,3); A_2=3; A_3=5$
(G)	$A_1=(2,3); A_2=3; A_3=(5,6)$	(H)	$A_1=2$ or $(2,3); A_2=(3,4); A_3=4$
(I)	$A_1=2$ or $(2,3); A_2=(3,4); A_3=(4,5)$	(J)	$A_1=2$ or $(2,3); A_2=(3,4); A_3=5$
(K)	$A_1=2$ or $(2,3); A_2=(3,4); A_3=(5,6)$		

Case (C) is considered in Subsubsection 6.3.2, case (F) is considered in Subsubsection 6.3.4. The other cases are considered in Subsubsection 6.3.3.

If the triple or quadruple of classes $\Omega_0(r_j)$ is a special case, then we use **2.** and **4.** and prove part 1) of the theorem for all almost special cases stemming from it – this will imply that part 1) holds for the initial triple of nilpotent orbits, see Subsubsection 6.3.6. For the almost special cases c1), d1) and d2) we prove that 2) holds as well, see Subsubsection 6.3.5.

Finally, we consider all cases obtained from one of the almost special ones a1), b1), c2) and d3) by an operation (s,l) on one of the three orbits (call such cases *neighbouring*), see Subsubsection 6.3.7. We prove that part 2) of the theorem holds for all these cases. Hence, part 2) will hold for all cases when $p=2$ or 3, when $r_1+\ldots+r_{p+1}=2n$ and when all special and almost special cases are avoided. Hence, it will hold when $r_1+\ldots+r_{p+1}\geq 2n$ and when all special and almost special cases are avoided.

6.3.2 Proof of the theorem for $p=2$, in case (C)

We construct triples of matrices satisfying the requirements of the lemma. The construction is done by induction on n. The induction base are the examples from Subsection 6.2.

The size n must be >3. Decrease the size of the matrices by 3 and delete a block of size 3 from each Jordan normal form. Denote the new Jordan normal forms by J_j'. This can lead only to case (C) again or to case (D) (with n replaced by $n-3$). By inductive assumption, there exists an irreducible triple of nilpotent matrices A_j' of size $n-3$ with $J(A_j')=J_j'$ and such that the matrix $B'=\alpha_1A_1'+\alpha_2A_2'+\alpha_3A_3'$ have a simple non-zero eigenvalue.

Denote by A_j'' the triple of matrices A_j from example (ex2). One can assume that the matrices B' and $B''=\alpha_1A_1''+\alpha_2A_2''+\alpha_3A_3''$ have no eigenvalue in common (to achieve this one can, if necessary, multiply one of the triples by $h\in\mathbf{C}^*$). By Lemma 36 (the reader should check that it is applicable), there exist irreducible triples of nilpotent matrices A_j with $J(A_j)=J_j$, the matrix B having at least two simple non-zero eigenvalues (hence, at least one). This proves the induction step in case (C).

6.3.3 Proof of the theorem for $p=2$, in cases (A), (B), (D), (E) and (G) – (K)

1^0. The cases (B), (D), (E), (G) and (H) – (K) are considered by analogy with (C) and we define only the matrices A_j''. As we make use of examples (ex3) – (ex7), one should keep in mind Lemma 39. Lemma 36 is applicable in all cases because there holds Lemma 40. (One can represent the triple of Jordan normal forms as a direct sum of triples of Jordan normal forms corresponding to one of the examples (ex1) – (ex7) and then use Lemma 40.)

In case (B) $J(A_3)$ must contain a block of size $\kappa\geq 5$. The matrices A_j'' of size κ are defined by examples (ex7) and (ex3) depending on whether n is even or odd. If $J(A_3)$ consists of a single block, then these examples define directly the matrices A_j. If not, then the matrices A_j' correspond again to case (B).

In case (D) the matrices A''_j are of size 12; they are the triple from example (ex6) (if $n = 12$, then case (D) is proved directly by example (ex6)). The matrices A'_j are from case (D) again.

In case (E) the matrices A''_j are of size 9; they are defined in example (ex4). The matrices A'_j are from case (E) or from case (D) or from case (F).

In case (G) there are at least three blocks of size 2 in $J(A_1)$. The matrices A''_j are defined by example (ex3). The matrices A'_j are either from case (G) or from case (F).

In cases (H) and (I) the matrices A''_j are of size 4; they are defined by example (ex1) with $n = 4$. One has to prove that there are at least two Jordan blocks of size 2 in $J(A_1)$, see [Ko2]. In case (H) the matrices A'_j are from case (H) or from case (D), in case (I) they are from case (I), (J), (H), (E), (F) or (D).

In case (J) the matrices A''_j are of size 10; they are defined by example (ex5) ($J(A_2)$ contains at least five blocks of size 3, otherwise $r_1 + r_2 + r_3 > 2n$). The matrices A'_j are either from case (J) or from case (F).

In case (K) there are at least three blocks of size 2 in $J(A_1)$ and at least two blocks of size 3 in $J(A_2)$ for the same reason, see [Ko2], and the matrices A''_j are of size 6, defined by example (ex3). The matrices A'_j are from case (K), (J), (G) or (F).

In all these cases the triple of Jordan normal forms of the matrices A'_j corresponds to neither of the special or almost special cases.

2^0. Consider case (A) (we follow the same ideas as in [Ko2]). Decrease by 1 the sizes of the matrices and decrease by 1 the sizes of two Jordan blocks respectively of $J(A_2)$ and $J(A_3)$. Delete a Jordan block of size 1 from $J(A_1)$. One can choose the diminished blocks of $J(A_2)$, $J(A_3)$ such that the triple of Jordan normal forms of size $n-1$ obtained like this not to correspond to any of the special or almost special cases. (It suffices to leave in each of the two Jordan normal forms of size $n-1$ a couple of blocks of different size.) This defines the Jordan normal forms of the matrices A'_j. The sum of their ranks equals $2(n-1)$.

Set $A^1_j = \begin{pmatrix} A'_j & G_j \\ 0 & A''_j \end{pmatrix}$. The matrices A''_j are of size 1 and equal 0. One constructs the blocks G_j such that the Jordan normal form of A^1_2 to be the necessary one (i.e. $\Omega_0(r_2)$). One sets $G_3 = 0$, $G_2 = -G_1$. (The condition $G_3 = 0$ can be achieved by conjugating the triple.) Hence, the matrices A^1_j *do not* define a semi-direct sum of L' and L'' because $J(A^1_2)$ is not a direct sum of $J(A'_2)$ and $J(A''_2)$.

Thus the matrix B^1 has a simple non-zero eigenvalue. It is an eigenvalue of B'.

After this one deforms the triple of matrices A^1_j into an irreducible one, with the necessary Jordan normal forms, see [Ko2]. One can also use the same reasoning as the one from case (F), see the next subsubsection. Hence, for the deformed triple the analog of the matrix B^1 still has a simple non-zero eigenvalue.

6.3.4 Proof of the theorem for $p = 2$, in case (F)

There exists an irreducible triple of nilpotent 9×9 matrices A'_1, A'_2, A'_3, $A'_1 + A'_2 + A'_3 = 0$ where $J(A'_1)$ ($J(A'_2)$; $J(A'_3)$) contains 1 block 3×3 and 3 blocks 2×2 (3 blocks 3×3; 1 block 4×4 and 1 block 5×5), see case (E). Hence, the matrix B' has a non-zero simple eigenvalue.

There exists a triple of nilpotent 15×15-matrices A^0_1, A^0_2, A^0_3, $A^0_1 + A^0_2 + A^0_3 = 0$ with $J(A^0_j) = \mathcal{J}_j$ where $\mathcal{J}_1$ ($\mathcal{J}_2$; $\mathcal{J}_3$) consists of 1 block 3×3 and 6 blocks 2×2 (of 5 blocks 3×3; of 3 blocks 5×5) and the matrices A^0_1, A^0_2 look like this (recall that A'_j are 9×9):

$$\left(\begin{array}{ccccccc} A_1' & \eta_1 & \eta_2 & \eta_3 & \eta_4 & \eta_5 & \eta_6 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) , \left(\begin{array}{ccccccc} A_2' & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

The matrix B^0 has the same non-zero eigenvalues as B'. Hence, it has a simple non-zero eigenvalue. One can deform the triple of matrices A_j^0 into an irreducible triple of matrices A_j with the necessary Jordan normal forms, see [Ko2]. Hence, the matrix B for small values of the deformation parameter still has a simple non-zero eigenvalue.

We explain the details of this deformation to show why this method is not applicable to some of the almost special cases (as was claimed in [Ko2]).

One looks for matrices A_j of the form $A_1 = A_1^0 + \varepsilon L$ where only the last row of L is non-zero, with $L_{15,14} = 1$; we assume that $L_{15,j} = 0$ for $j = 10, 11, 12, 13, 15$ and that for $\varepsilon \neq 0$ the sizes of the Jordan blocks of $J(A_1)$ equal 3,2,2,2,2,2,2; one sets $A_j = (I + \varepsilon X_j(\varepsilon))^{-1} A_j^*(I + \varepsilon X_j(\varepsilon)$, $j = 2, 3$, where X_j are analytic in $\varepsilon \in (\mathbf{C}, 0)$ (their existence is justified by the basic technical tool).

The matrix algebra $\mathcal{A}^0$ generated by the matrices A_j contains matrices of the form $\left(\begin{array}{cc} P & Q \\ O(\varepsilon) & O(\varepsilon) \end{array}\right)$ for all P, Q where $P \in gl(9, \mathbf{C})$. The centralizer of $\mathcal{A}^0$ is trivial, see [Ko2].

Fix a matrix $F \in \mathcal{A}^0$ with $P = I$, $Q = 0$. By taking powers of F, one can assume that the 6 right columns of F are identically 0.

Conjugate the matrices A_j by a matrix $\left(\begin{array}{cc} I & 0 \\ O(\varepsilon) & I \end{array}\right)$ so that after the conjugation the matrix F become equal to $\left(\begin{array}{cc} I & 0 \\ 0 & 0 \end{array}\right)$. Hence, $\mathcal{A}^0$ contains all matrices of the form $Y = \left(\begin{array}{cc} P & Q \\ 0 & 0 \end{array}\right)$ (if $S \in \mathcal{A}^0$, then $Y = FS \in \mathcal{A}^0$). Two cases are possible after the conjugation:

1) the three matrices become block upper-triangular, with diagonal blocks of sizes 9 and 6;

2) there exists an entry in the left lower block 6×9 which is $\neq 0$ for $\varepsilon \neq 0$.

Eliminate case 1). If this were true, then the restrictions of A_j to the right lower block 6×6 would be nilpotent, with sizes of the blocks equal to 2,2,2; 3,3; 5,1. They must define a nilpotent algebra $\mathcal{A}^1$.

Indeed, it is impossible to have an irreducible triple with such sizes of the blocks because one would have $r_1 + r_2 + r_3 = 11 < 12$. It is impossible to conjugate the algebra to a block upper-triangular form with at least one diagonal block $\tilde{P}$ irreducible, of size $m > 1$. Indeed, the restrictions of the matrices to such a block would be nilpotent, with sizes of the blocks not greater respectively than 2, 3 and 5. One checks directly that for $m = 2, 3, 4$ and 5 it is impossible to have $r_1' + r_2' + r_3' \geq 2m$, r_j' being the ranks of the restrictions of the matrices to the block $\tilde{P}$. (It is this part of the reasoning which is not applicable to the proof of the almost special cases a1), b1), c2) and d3); e.g., in case b1) there exist block upper-triangular triples 6×6 with irreducible diagonal blocks 3×3.)

On the other hand if $\mathcal{A}^1$ is nilpotent, then it can be conjugated to an upper-triangular form. The restriction to the right lower block 6×6 of the matrix $(A_1 + A_2/2)|_{\varepsilon=0}$ has non-zero entries in positions $(k, k+1)$, $k = 10, \ldots, 14$. Hence, the conjugation can be carried out by a matrix

$I+O(\varepsilon)$. Such a conjugation cannot annihilate the entry $A_{1;15,14}$. Hence, the algebra $\mathcal{A}^1$ is not nilpotent.

Consider case 2). Let the algebra $\mathcal{A}^0$ contain a matrix S with $S_{10,j}|_{\varepsilon=0}\neq 0$ for some $j\leq 9$ (if one has $S_{i,j}|_{\varepsilon=0}\neq 0$ for $i>10$ and for some $j\leq 9$, then one can multiply S by $(A_1+A_2/2)^{i-10}$ to have $S_{10,j}|_{\varepsilon=0}\neq 0$).

Then one can assume that only the left lower block 6×9 of S is non-zero (one can consider instead of S the matrix $SF-FSF$). Multiply the matrix S by matrices Y defined above. Hence, $\mathcal{A}^0$ contains matrices of the form $\left(\begin{array}{cc} P & Q \\ O(\varepsilon) & O(\varepsilon)\end{array}\right)$ with $P\in gl(10,\mathbf{C})$, for all P and Q and one can repeat the reasoning which led us to cases 1) and 2), but this time the size of the block P has increased by 1. Continuing like this, we see that $\mathcal{A}^0=gl(15,\mathbf{C})$, i.e. the triple A_1, A_2, A_3 is irreducible.

6.3.5 Proof of the theorem in the almost special cases c1), d1) and d2)

There exist irreducible triples of nilpotent matrices A_j satisfying (2) with sizes of the Jordan blocks like in cases c1), d1) or d2) but with $g=1$. These triples can be obtained by deforming respectively the ones from examples (ex1) with $n=4$ for case c1) and (ex3) for cases d1) and d2).

Consider the direct sum of such a triple and of a triple from example (ex1) with $n=4$ for case c1) and of one from example (ex3) for cases d1) and d2). Lemma 36 is applicable to such a direct sum which provides the existence of irreducible triples from cases c1), d1) and d2) for $g=2$. In the same way one constructs such triples for all $g>1$ – by deforming the direct sum of a triple for $g-1$ and of one from example (ex1) with $n=4$ for case c1) or (ex3) for cases d1) and d2) and by using Lemma 36.

The irreducible representations thus obtained can be considered as deformations of certain direct or semi-direct sums of representations whose diagonal blocks are of sizes 4 or 6 and whose matrices B have distinct non-zero eigenvalues, see Lemma 39. This property persists under small deformations.

6.3.6 Proof of the theorem in the almost special cases a1), b1), c2) and d3)

We consider only case b1) in detail. The other cases are treated by analogy and we explain the differences at the end of the subsubsection. Recall that in all these cases we prove the existence of nice representations.

Construct the matrices $A_j=\left(\begin{array}{ccccc} A_j^1 & 0 & \ldots & 0 & H_j^1 \\ 0 & A_j^2 & \ldots & 0 & H_j^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & A_j^{g-1} & H_j^{g-1} \\ 0 & 0 & \ldots & 0 & A_j^g\end{array}\right)$ where A_j^k are 3×3, nilpotent, of rank 2 and the representations defined by the triples A_j^k are irreducible for all k. Moreover, they are presumed to be non-equivalent (this can be achieved by multiplying them by constants $g_k\in\mathbf{C}^*$) and the matrices B^k to have non-zero distinct eigenvalues; the eigenvalues of the matrix B can be presumed non-zero and distinct as well; see example (ex2).

Assume that the matrix A_1 is in upper-triangular Jordan normal form. Then we set $H_1^k=\left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1\end{array}\right)$ for all k. Hence, the Jordan normal form of A_1 consists of $g-2$ blocks of size 3,

of a block of size 4 and of a block of size 2 (to be checked directly).

The blocks H_j^k for $j=2,3$ are defined such that $H_j^k = A_j^k D_j^k - D_j^k A_j^d$, $D_j^k \in gl(3,\mathbf{C})$ and $H_1^k + H_2^k + H_3^k = 0$. Such a choice of H_j^k is possible because the representations defined by the triples A_j^k for different values of k are irreducible non-equivalent and the mapping

$$(D_2^k, D_3^k) \mapsto A_2^k D_2^k - D_2^k A_2^g + A_3^k D_3^k - D_3^k A_3^g$$

is surjective onto $gl(3,\mathbf{C})$. Notice that the blocks H_2^k, H_3^k result from conjugation of A_j with $I + D_j$ where D_j has non-zero entries only in the last 3 columns and first $3(g-1)$ rows (its restriction to the rows with indices $3k-2$, $3k-1$, $3k$ and to the last three columns equals D_j^k). Hence, $J(A_2)$ and $J(A_3)$ consist each of g blocks of size 3.

Prove that the centralizer $\mathcal{Z}$ of the triple of matrices A_j is trivial. Block-decompose a matrix from $gl(n,\mathbf{C})$ into blocks 3×3 (denoted by $X^{\mu,\nu}$). For $Y \in \mathcal{Z}$ set $Y^{\mu,\nu} = Y|_{X^{\mu,\nu}}$. One has (first for $\nu < \mu = g$ and then for $\mu \geq \nu$, $\nu \leq g-1$) $Y^{\mu,\nu} A_j^\nu - A_j^\mu Y^{\mu,\nu} = 0$. Hence, if $\mu \neq \nu < g$, then $Y^{\mu,\nu} = 0$, if $\mu = \nu$, then $Y^{\mu,\nu} = \alpha_\mu I$ (we use the non-equivalence of the representations and Schur's lemma).

For $\nu = g$ one has $(\alpha_k - \alpha_g) H_1^k + A_1^k Y^{k,g} - Y^{k,g} A_1^g = 0$. Hence, $\alpha_k = \alpha_g$ because H_1^k is not of the form $A_1^k Y^{k,g} - Y^{k,g} A_1^g$. But then $A_j^k Y^{k,g} - Y^{k,g} A_j^g = 0$ for $j=2,3$ which implies $Y^{k,g} = 0$ for $k < g$. Hence, the centralizer is trivial.

In all other almost special cases one similarly constructs triples or quadruples of matrices A_j satisfying the conclusions from 1) of the theorem.

In all cases the matrix A_j whose Jordan form has to be changed (from equal sizes of the Jordan blocks to one obtained by replacing a couple l_j, l_j of sizes by l_j+1, l_j-1) has equal blocks H_j^k which have a unit in the right lower corner and zeros elsewhere; the restriction of the matrix A_j to the diagonal blocks is in Jordan normal form. The diagonal blocks are of sizes 2, 4 or 6. We leave the details for the reader.

6.3.7 Proof of the theorem in the neighbouring cases

There are two types of neighbouring cases. Recall that an almost special case is obtained from a special one by an operation (s,l) performed on one of the three or four Jordan normal forms and a neighbouring case is obtained by performing another such operation. In the first type the second operation is performed on one of the other Jordan normal forms, in the second type it is performed on the same one. We consider only cases neighbouring to a1), b1), c2) and d3), in the cases neighbouring to c1), d1) and d2) there exist irreducible representations, see Subsubsection 6.3.5.

Remark: In all neighbouring cases the triples or quadruples can be considered as deformations of triples or quadruples from almost special cases, in which the matrix B has distinct non-zero eigenvalues. Hence, this is so in all neighbouring cases as well.

Neighbouring cases of the first type.

Construct an irreducible triple or quadruple in a neighbouring case of the first type. Let A_j^* be the matrices from the triple or quadruple of the corresponding almost special case as constructed in the previous subsubsection.

Consider the only case of first type neighbouring to b1). (It is the only one up to permutation of the three Jordan normal forms. The rest of neighbouring cases of the first type are considered by analogy.) Suppose that it is the orbit of A_2^* to be changed and that A_2^* is in upper-triangular Jordan normal form. We assume that the triple A_1^*, A_2^*, A_3^* is obtained from the triple of matrices A_j constructed in the previous subsubsection by conjugation with $(I+D_2)^{-1}$. Set

$A_2 = A_2^* + \varepsilon A_2^0$ where the matrix A_2^0 has non-zero entries only in the last three rows and first $n-3$ columns. The restriction of A_2^0 to the block $X^{g,k}$ defined in the previous subsubsection equals H_1^k (defined also there). Hence, for $\varepsilon \neq 0$ the Jordan normal form of A_2 is the required one.

After this look for A_j $(j=1,3)$ in the form $A_j = (I+\varepsilon X_j(\varepsilon))^{-1}A_j^*(I+\varepsilon X_j(\varepsilon)$ where X_j are analytic in $\varepsilon \in (\mathbf{C},0)$ (their existence is justified by the basic technical tool).

Show that the algebra $\mathcal{A}'$ generated by the matrices A_j is $gl(n,\mathbf{C})$ from where part 2) of the theorem follows. The algebra $\mathcal{A}$ generated by the matrices A_j^* is the one of all matrices W having arbitrary entries in the diagonal blocks $X^{i,i}$ and in the blocks $X^{g,i}$. This follows from the basic result in [Ko4]. Hence, $\mathcal{A}'$ contains matrices of the form $S+\varepsilon T$ for all $S\in\mathcal{A}$.

The algebra $\mathcal{A}'$ contains the matrix $(A_2)^3/\varepsilon = \sum_{\kappa=1}^{g-1} E_{3g-2,3\kappa}$. By multiplying and postmultiplying it by matrices from $\mathcal{A}'$, one can obtain matrices of the form $V+\varepsilon Z$ for V having any restriction to the block $X^{i,j}$, for all i and for $j\leq g-1$.

The matrices W and V contain a basis of $gl(n,\mathbf{C})$ for $\varepsilon\neq 0$ small enough. Hence, the triple A_1, A_2, A_3 is irreducible.

Neighbouring cases of the second type.

1^0**.** The cases of second type neighbouring to a given almost special case can be characterized by the sizes of the blocks of the Jordan normal form which changes w.r.t. the corresponding special case. There are four possibilities:

1) $l_j+2, l_j-2, l_j, \ldots, l_j$	2) $l_j+2, l_j-1, l_j-1, l_j, \ldots, l_j$
3) $l_j+1, l_j+1, l_j-2, l_j, \ldots, l_j$	4) $l_j+1, l_j+1, l_j-1, l_j-1, l_j, \ldots, l_j$

Possibilities 1), 2), 3) and 4) appear for the first time respectively for $g=2$, $g=3$, $g=3$ and $g=4$. We explain the construction for these minimal values of g, for all others the existence is proved by induction on g, when considering direct sums of triples or quadruples constructed for $g-1$ and triples or quadruples defined by examples (ex0), (ex1) with $n=4$ or (ex3). We deform such direct sums into irreducible representations by means of Lemma 36.

2^0**. Possibility 1).**

Let $g=2$. Explain in details the case neighbouring to b1). Denote by $A_j^* = \begin{pmatrix} A_j^1 & H_j^1 \\ 0 & A_j^2 \end{pmatrix}$ matrices defining triples from case b1) with $g=2$ and H_j^1 defined like in the previous subsubsection. The centralizer of the triple is trivial and the representations defined by the matrices A_j^1 and A_j^2 are non-equivalent. A_1^1 and A_1^2 are upper-triangular Jordan blocks of size 3. Hence, the matrix algebra $\mathcal{A}$ generated by the matrices A_j^* contains all block upper-triangular matrices with blocks 3×3, see [Ko4]. In particular, it contains the matrix $S = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$.

Set $A_1 = A_1^* + \varepsilon Y$, $Y = E_{4,1}$. Hence, A_1 has for $\varepsilon\neq 0$ Jordan blocks of sizes 5 and 1. Set $A_j = (I+\varepsilon X_j(\varepsilon))^{-1}A_j^*(I+\varepsilon X_j(\varepsilon))$, X_j being analytic in $\varepsilon\in(\mathbf{C},0)$ and such that $\sum_j A_j = 0$ (the existence of X_j follows from the basic technical tool).

The algebra $\mathcal{A}'$ contains for all $F\in\mathcal{A}$ a matrix $F+O(\varepsilon)$. Hence, $\mathcal{A}'$ contains a matrix $S' = S+O(\varepsilon)$. Hence, it contains the matrix $Q = S'(A_1)^3$ which is of the form $\varepsilon\begin{pmatrix} * & * \\ G & * \end{pmatrix} + o(\varepsilon)$ with $G = E_{4,3}\neq 0$. By multiplying and postmultiplying Q/ε by matrices from $\mathcal{A}'$, one can obtain matrices of the form of Q/ε with any block G. These matrices together with the matrices $F+O(\varepsilon)$ form a basis of $gl(6,\mathbf{C})$. Hence, the triple of matrices A_j is irreducible.

In all other neighbouring cases with possibility 1) the left lower block of the matrix Y has a single unit in its left upper corner and zeros elsewhere.

3^0. **Possibility 2).**

Consider the case neighbouring to b1) (the ones neighbouring to a1), c2) and d3) are considered by analogy). Consider a block upper-triangular triple of matrices A_j^1 with a trivial centralizer like in case b1) with $g = 2$; the diagonal blocks are 3×3, they define non-equivalent representations. Consider its direct sum with an irreducible triple of matrices A_j^2 defined by example (ex2). There exists a semi-direct sum of such triples (one can use arguments like the ones from the proof of Lemma 36; the matrices are block upper-triangular, with diagonal blocks 3×3). After this one deforms the triple into a nearby irreducible one like in the previous example – one sets

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $A_j = (I + \varepsilon X_j(\varepsilon))^{-1} A_j^*(I + \varepsilon X_j(\varepsilon))$ for $j = 2, 3$. The matrix algebra $\mathcal{A}$ contains all block upper-triangular matrices from $gl(9, \mathbf{C})$ with blocks 3×3, see [Ko4]. Hence, $\mathcal{A}'$ contains a matrix of the form $F = E_{7,7} + O(\varepsilon)$.

One has $(A_1)^3 = E_{1,4} + \varepsilon E_{7,3}$. The matrix $R = F(A_1)^3/\varepsilon$ belongs to $\mathcal{A}'$. One has $R_{7,3} \neq 0$ for $\varepsilon = 0$. Like in the previously considered case one concludes that $\mathcal{A}' = gl(9, \mathbf{C})$ and that the triple A_1, A_2, A_3 is irreducible.

4^0. **Possibility 3).**

In the case neighbouring to b1) (the ones neighbouring to a1), c2) and d3) are considered by analogy) one has $g \geq 3$. For $g = 3$ one sets $A_j = A_j^* + \varepsilon A_j^0$ where the triple of matrices A_j^* is block upper-triangular, with blocks 3×3, the diagonal blocks defining non-equivalent irreducible representations. Set

$$A_1^* + \varepsilon A_1^0 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \varepsilon & 0 & 0 & -\varepsilon & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For $j = 2, 3$ one sets $A_j^* = \begin{pmatrix} A_j^1 & 0 & A_j^1 D_j^1 - D_j^1 A_j^3 \\ 0 & A_j^2 & A_j^2 D_j^2 - D_j^2 A_j^3 \\ 0 & 0 & A_j^3 \end{pmatrix}$ (i.e. A_j^* are the matrices A_j from the previous subsubsection). The blocks A_j^i, $i = 1, 2, 3$ are nilpotent rank 2 matrices. Set $A_j = (I + \varepsilon X_j(\varepsilon))^{-1} A_j^*(I + \varepsilon X_j(\varepsilon))$ for $j = 2, 3$. We let the reader check oneself that the matrices A_j have the necessary Jordan normal forms.

One shows next that the algebra $\mathcal{A}$ contains all matrices with arbitrary blocks in positions (1,1), (1,3), (2,2), (2,3), (3,3) (using [Ko4]). Hence, the algebra $\mathcal{A}'$ contains a matrix $P =$

$E_{7,7}+O(\varepsilon)$ and the matrix $P(A_1)^3/\varepsilon$ has (for $\varepsilon=0$) non-zero entries in positions (7,3) and (7,6). Like in the previous case one conclude that $\mathcal{A}'=gl(9,\mathbf{C})$, i.e. the triple A_1, A_2, A_3 is irreducible.

5^0**. Possibility 4).**

One must have $g\geq 4$. Let $g=4$. Consider the case neighbouring to b1). One constructs a triple of matrices A_j^* with

$$A_1^*=\left(\begin{array}{cccc}A_1^0&\varphi_1&\varphi_2&\varphi_3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{array}\right)~,~A_2^*=\left(\begin{array}{cccc}A_2^0&\eta_1&\eta_2&\eta_3\\0&0&1&0\\0&0&0&1\\0&0&0&0\end{array}\right)$$

and $A_3^*=-A_1^*-A_2^*$, A_j^0 are 9×9. The blocks of $J(A_j^0)$ are of sizes 4,4,1; 3,3,3; 3,3,3, the triple of nilpotent matrices A_j^0 is irreducible. Its existence follows from the case neighbouring to b1) from possibility 3). One chooses the vector-columns φ_j and η_j such that $J(A_2^*)$ and $J(A_3^*)$ to have each four blocks of size 3 and $J(A_1^*)$ to have blocks of sizes 4,4,2,1,1.

Assume that A_1^* is in Jordan normal form and that φ_1 has a unit in its last position and zeros elsewhere. Set $A_1=A_1^*+\varepsilon L$ where only the last row of L is non-zero, with $L_{12,11}\neq 0$ and $L_{12,10}=L_{12,12}=0$. One chooses L such that for $\varepsilon\neq 0$ the sizes of the blocks of $J(A_1)$ to be 4,4,2,2. Set for $j=2,3$ $A_j=(I+\varepsilon X_j(\varepsilon))^{-1}A_j^*(I+\varepsilon X_j(\varepsilon))$. After this the irreducibility of the triple A_1, A_2, A_3 is proved by analogy with case (F).

In the case neighbouring to c2) one sets

$$A_1^*=\left(\begin{array}{ccccc}A_1^0&\varphi_1&\varphi_2&\varphi_3&\varphi_4\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\end{array}\right)~,~A_2^*=\left(\begin{array}{ccccc}A_2^0&\eta_1&\eta_2&\eta_3&\eta_4\\0&0&-1&0&0\\0&0&0&-1&0\\0&0&0&0&-1\\0&0&0&0&0\end{array}\right)$$

where $A_j^0\in gl(12,\mathbf{C})$. The sizes of the blocks of $J(A_j^*)$ are 5,5,3,3; 4,4,4,4; 2,2,2,2,2,2,2,1,1. The sizes of the blocks of $J(A_j^0)$ are 5,5,2; 4,4,4; 2,2,2,2,2,2. The existence of such an irreducible triple of nilpotent matrices A_j^0 follows from the case neighbouring to b1) from possibility 3). Set $A_3=A_3^*+\varepsilon L$, $A_j=(I+\varepsilon X_j(\varepsilon))^{-1}A_j^*(I+\varepsilon X_j(\varepsilon))$, $j=1,2$. One has $L_{16,15}\neq 0$ and $L_{16,13}=L_{16,14}=L_{16,16}=0$. For the rest of the reasoning is like in the previous case.

In the case neighbouring to d3) one sets

$$A_1^*=\left(\begin{array}{cc}A_1^0&T_1\\0&A_1'\end{array}\right)~,~A_2^*=\left(\begin{array}{cc}A_2^0&T_2\\0&A_2'\end{array}\right)\qquad(18)$$

where $A_j^0\in gl(18,\mathbf{C})$ and

$$A_1'=\left(\begin{array}{cccccc}0&0&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\\0&0&0&0&0&0\end{array}\right)~,~A_2'=\left(\begin{array}{cccccc}0&-1&0&0&0&0\\0&0&-1&0&0&0\\0&0&0&0&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&-1\\0&0&0&0&0&0\end{array}\right)~;$$

$J(A_2^*)$ has blocks of size 3; $J(A_3^*)$ has eleven blocks of size 2 and two of size 1; $J(A_1^*)$ has blocks of sizes 7,7,5,5. The sizes of the blocks of $J(A_j^0)$ are respectively 7,7,4; six times 3; nine times 2.

Set $A_3 = A_3^* + \varepsilon L$, $A_j = (I + \varepsilon X_j(\varepsilon))^{-1} A_j^* (I + \varepsilon X_j(\varepsilon))$, $j = 1, 2$. One has $L_{24,23} \neq 0$ and $L_{24,j_0} = 0$ for $1 \leq j_0 \leq 24$, $j_0 \neq 23$. Irreducible triples of such nilpotent matrices A_j^0 exist by the case neighbouring to d3) from possibility 3). The rest of the reasoning is like in the previous two cases.

In the case neighbouring to a1) one represents A_j^* in the form (18) with

$$A_1' = A_4' = 0 \ , \ A_2' = -A_3' = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with $A_j^0 \in gl(6, \mathbf{C})$. The sizes of the blocks of $J(A_j^*)$ are 3,3,1,1; 2,2,2,2; 2,2,2,2; 2,2,2,1,1. The ones of $J(A_j^0)$ are 3,3; 2,2,2; 2,2,2; 2,2,2. The existence of an irreducible quadruple of nilpotent matrices A_j^0 follows from the case neighbouring to a1) from possibility 3). Set $A_4 = A_4^* + \varepsilon L$, $A_j = (I + \varepsilon X_j(\varepsilon))^{-1} A_j^* (I + \varepsilon X_j(\varepsilon))$, $j = 1, 2, 3$. The matrix L has a single non-zero entry in position $(8, 7)$. The rest of the reasoning is like in the previous three cases.

The theorem is proved.

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